4_29 Examples of the Riemann integral

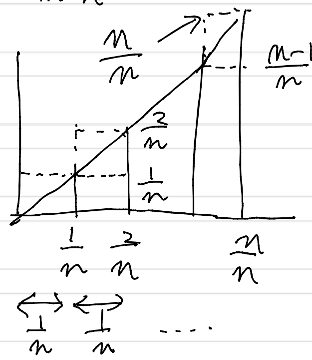
Example 0 (An additional example (not in the text))

$I=[0,1]$, $f(x)=x$

$P_n=\{I_k^{(n)}\}_{k=1}^{n}$ $\quad I_k^{(n)}=\left[\frac{k-1}{n},\frac{k}{n}\right]$

$$\lim_{n\to\infty}L(f,P_n)=\lim_{n\to\infty}U(f,P_n)=\sup_{P\in\Pi}L(f,P)=\inf_{P\in\Pi}U(f,P)$$

(Known fact)



$$L(f,P_n)=\frac{0}{n^2}+\frac{1}{n^2}+\cdots+\frac{n-1}{n^2}$$

$$=\frac{1}{n^2}\sum_{k=0}^{n-1}k=\frac{1}{n^2}\frac{(n-1)n}{2}$$

$$U(f,P_n)=\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}$$

$$=\frac{1}{n^2}\sum_{k=1}^{n}k=\frac{1}{n^2}\frac{n(n+1)}{2}$$

$$\lim_{n\to\infty}L(f,P_n)=\lim_{n\to\infty}\frac{n^2-n}{2n^2}=\lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{2n}\right)=\frac{1}{2}$$

$$\lim_{n\to\infty}U(f,P_n)=\lim_{n\to\infty}\frac{n^2+n}{2n^2}=\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right)=\frac{1}{2}$$

$$L(f,P_{n+1})-L(f,P_n)=\frac{n(n+1)}{2(n+1)^2}-\frac{n(n-1)}{2n^2}$$

$$=\frac{n^3(n+1)-n(n+1)^2(n-1)}{2n^2(n+1)^2}=\frac{n^2-(n+1)(n-1)}{2n(n+1)}$$

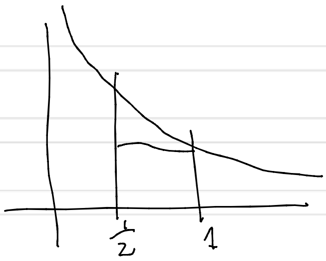
$$=\frac{1}{2n(n+1)}>0 \text{ (increasing)}$$

$$U(f,P_{n+1})-U(f,P_n)=\frac{(n+1)(n+2)}{2(n+1)^2}-\frac{n(n+1)}{2n^2}$$

$$=\frac{n(n+2)-(n+1)^2}{2(n+1)^2n}=\frac{-1}{2(n+1)^2n}<0$$

decreasing

Example 1.4 $f = \begin{cases} \frac{1}{x} & x \in (0, 1] \\ 0 & x = 0 \end{cases}$



f : is not bounded

$\forall a > 0$ (even if it's near to 0)

$\sup_{x \in [0,a]} f(x) = \infty$

$U(f,P)$: can not be defined.

And the fact

$$\sup_{P} L(f,P) = \infty$$

Usually $\int_0^1 f(x)\,dx$ is defined by

$$\int_0^1 f(x)\,dx = \lim_{a \downarrow 0} \int_a^1 f(x)\,dx$$

· $f(x)$ is bounded (continuos) function on $[a, 1]$, $0 < \int_a^1 f(x)\,dx < \infty$

$$\int_1^\infty f(x)\,dx = \lim_{M \to \infty} \int_1^M f(x)\,dx = \infty$$

$$\Big(\int_a^1 \frac{1}{x}\,dx = -\log a = \log \frac{1}{a} \xrightarrow[a \downarrow 0]{} \infty$$

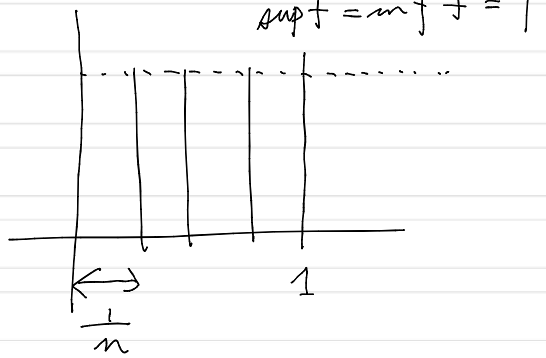
$$\int_1^M \frac{1}{x}\,dx = \log M \to \infty \quad M \to \infty \Big)$$

Example 1.5 $\quad f(x) = 1 \quad x \in [0,1]$

$$L(f, P_n) = U(f, P_n) = 1$$

$$U(f,P_n) = \sum_{k=1}^{n} 1 \times \frac{1}{n} = 1$$

$$\int_0^1 1\, dx = 1$$



$$\int_a^b \alpha \, dx = (b-a)\alpha$$

Example 1.7

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

$\mathbb{Q}$: all rational numbers

x: an irrational number in $[0,1]$

$|b-a| > 0$

$\forall a, b, (a < b)$

$\exists x, y \in (a,b), \quad x \in \mathbb{Q}, \ y \notin \mathbb{Q}$ (rational, irrational)

$$\Rightarrow \sup_{x \in [a,b]} f(x) = 1, \quad \inf_{x \in [a,b]} f(x) = 0$$

$$L(f,P) = \sum_{k=1}^{n} \Big(\underbrace{\inf_{x \in I_n} f(x)}_{=0}\Big) \times |I_n| = 0$$

$$U(f,P) = \sum_{k=1}^{n} \Big(\sup_{x \in I_n} f(x)\Big) \times |I_n| = |I| = 1$$

$$\sup L(f,P) \neq \inf U(f,P)$$

$$\sup L(f,P) = 0, \quad \inf U(f,P) = 1$$

f is not Riemann integrable

(2) $1+2+\cdots+n=?$

$S=1+2+\cdots+n$

$2S=(1+2+\cdots n)+(n+(n-1)+\cdots 1)$

$=(1+n)+(2+n-1)+\cdots+(n+1)$

$=n(n+1)$

$S=\frac{1}{2}n(n+1)$