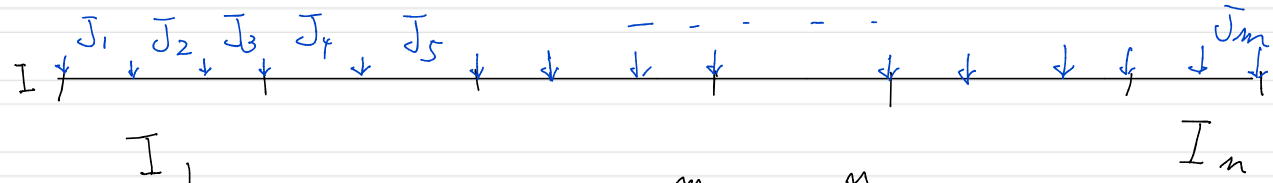
5/6 Refinement of partitions

Def 1.8 $Q=\{J_1,\cdots,J_m\}$, $P=\{I_1,\cdots,I_n\}$: partitions of I

Q is a refinement of P $\quad(n\le m)$

$\overset{\text{def}}{\iff}$ $\quad I_k=\bigcup_{l=1}^{L_k} J_{i_{k,l}}$, $\quad 1\le i_{k,l}\le m$

for some $\{J_{k,l}\}_{l=1}^{L_k}$ for each $k=1,\cdots,n$



$\{J_j\}$: almost disjoint $\qquad \bigcup_{j=1}^{m} J_j=\bigcup_{i=1}^{n} I_i=I$

※ Each I_i is divided by several intervals,

$I=\{[a_{i-1},a_i]\}_{i=1}^{n}$, $\quad J=\{[b_{j-1},b_j]\}_{j=1}^{m}$

$\;=\{a_i\}_{i=0}^{n}$ $\qquad\qquad\quad =\{b_j\}_{j=0}^{m}$

(These can be expressed by all start and end points.)

Increasing sequences of positive integers

J is a refinement of I $\iff$ $\{a_i\}\subset\{b_j\}$

Example 1.9, 1.10

$P=\{0,\frac{1}{2},1\}$, $Q=\{0,\frac{1}{3},\frac{2}{3},1\}$, $R=\{0,\frac{1}{4},\underset{=\frac{1}{2}}{\frac{2}{4}},\frac{3}{4},1\}$

R is a refinement of P.

Q is not a refinement of P.

$S=P\cup Q=\{0,\frac{1}{3},\frac{1}{2},\frac{2}{3},1\}$.

S is a refinement of P and Q.

In general, $P\cup Q$ is a refinement of both P and Q,

for any partitions P and Q.

$T=\{0,\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4},1\}$ is a refinement of P,Q,R,S

Theorem 1.11 $f : [a,b] \to \mathbb{R}$ bounded.

P, Q : partitions of $[a,b]$, Q is a refinement of P.

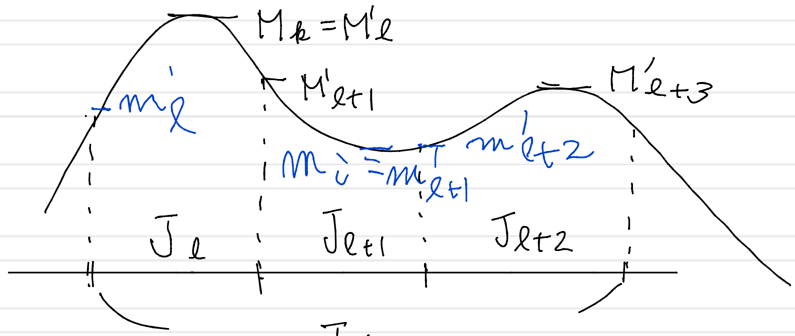
$$L(f,P) \leq L(f,Q), \quad U(f,Q) \leq U(f,P)$$

(Proof) Assume that the two partition P, Q are expressed by.

$$P = \{I_i\}_{i=1}^{n}, \quad Q = \{J_j\}_{j=1}^{m}$$

and that $I_k = \bigcup_{l=p_k}^{q_k} J_l$. $\quad (q_k + 1 = p_{k+1})$

$$\underbrace{\sup_{x \in I_k} f(x)}_{\shortparallel \ M_k} \geq \underbrace{\sup_{x \in J_l} f(x)}_{\shortparallel \ M'_l} \quad \text{for } l = p_k, \cdots, q_k$$



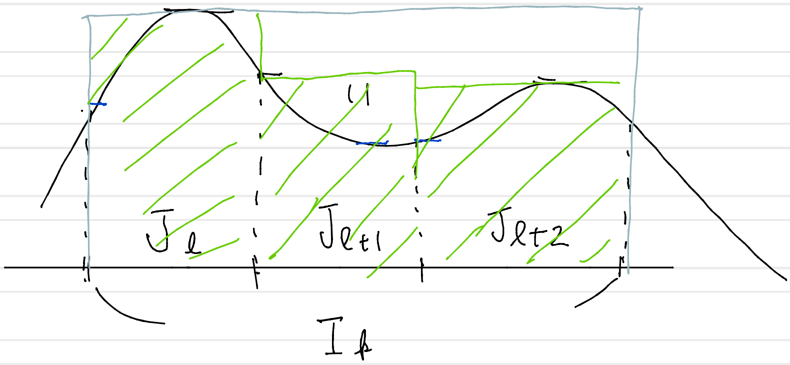
$$m_k = \inf_{x \in I_k} f(x), \quad m'_l = \inf_{x \in J_l} f(x)$$

$$U(f,P) = \sum_{k=1}^{n} \sup_{x \in I_k} f(x) |I_k|$$

$$= \sum_{k=1}^{n} \sum_{l=p_k}^{q_k} \overbrace{\sup_{x \in I_k} f(x)}^{M_k} |J_l|$$

$$\geq \sum_{k=1}^{n} \sum_{l=p_k}^{q_k} \overbrace{\sup_{x \in J_l} f(x)}^{M'_k} |J_l|$$

$$= \sum_{l=1}^{m} \sup_{x \in J_l} f(x) |J_l| = U(f,Q)$$



Using the same arguments we have;

$$L(f,P) = \sum_{k=1}^{n} \overbrace{\inf_{x\in I_k} f(x)}^{m_k} |I_k|$$

$$= \sum_{k=1}^{n} \inf_{x\in I_k} f(x) \sum_{\ell=p_k}^{q_k} |J_\ell|$$

$$\le \sum_{\ell=1}^{m} \underbrace{\inf_{x\in J_\ell} f(x)}_{m'_\ell} |J_\ell| = M(f,Q)$$

Prop. 1.12 $\quad L(f,P) \le U(f,P)$

(Proof) $\quad L(f,P) = \sum_{k=1}^{n} \inf_{x\in I_k} f(x) \, |I_k|$

$$\le \sum_{k=1}^{n} \sup_{x\in I_k} f(x) |I_k| = U(f,P)$$

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