5/13 Cauchy criterion for integrability.

$\{a_n\}$: a sequence of real numbers

$a_n \to \alpha \quad (\alpha = \lim_{n\to\infty} a_n)$

$\Leftrightarrow \left(\begin{array}{l} \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \text{ s.t.} \\ n \geq N \Rightarrow |a_n - \alpha| < \varepsilon \end{array} \right.$

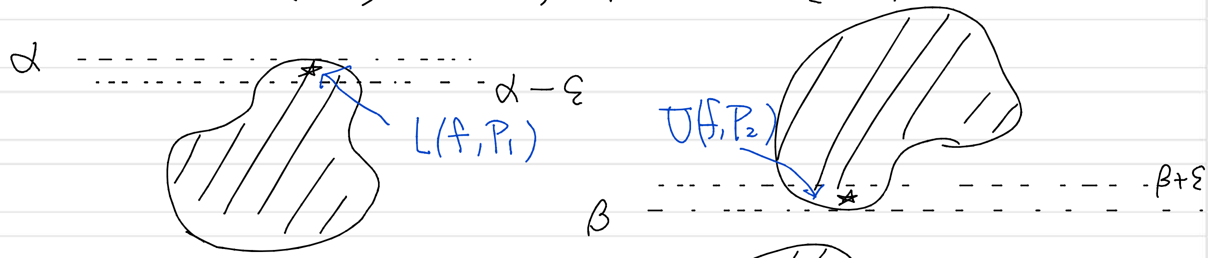
(For any difference (error) $\varepsilon > 0$,
$|a_n - \alpha|$ are less than ε for large enough n.

$\alpha = \sup_{P\in\Pi} L(f,P) \quad , \quad \beta = \inf_{P\in\Pi} \overline{U}(f,P) \quad$ (Π is the set of all partitions

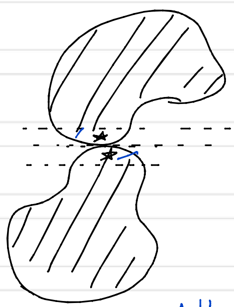
(P is an arbitrary partition of a given interval I.)

$\forall \varepsilon > 0 \quad , \quad \exists P_1, P_2 \in \Pi$

$\alpha - \varepsilon < L(f,P_1) \leq \alpha \quad , \quad \beta \leq \overline{U}(f,P_2) < \beta + \varepsilon$



※ f is integrable $\Leftrightarrow \alpha = \beta$



Theorem 1.14

f is Riemann integrable on $I = [a,b]$ $\Longleftrightarrow \left(\begin{array}{l} \forall \varepsilon > 0 \quad \exists P \in \Pi \text{ s.t.} \\ \overline{U}(f,P) - L(f,P) < \varepsilon \end{array} \right.$

All partitions of $I = [a,b]$ ↓ Π

(Proof)

$\Rightarrow$ f is Riemann integrable,

$\sup_P L(f,P) = \inf_P \overline{U}(f,P)$ (definition)

Let $\varepsilon > 0$ be an any positive number. We consider that $\frac{\varepsilon}{2}$ is an arbitrary number. Then, $\exists P_1, P_2 \in \Pi$, s.t.

$$\alpha - \frac{\varepsilon}{2} < L(f, P_1) \quad , \quad \overline{U}(f, P_2) < \alpha + \frac{\varepsilon}{2}$$

Let $P_3 = P_1 \cup P_2$ (P_3 is a refinement of P_1, P_2)

By Th. 1.11, $L(f, P_1) \le L(f, P_3)$, $\overline{U}(f, P_2) \ge \overline{U}(f, P_3)$

Then, we have:

$$\alpha - \frac{\varepsilon}{2} < L(f, P_1) \le L(f, P_3) \ , \ \overline{U}(f, P_3) \le \overline{U}(f, P_2) < \alpha + \frac{\varepsilon}{2}$$

$$\alpha = \sup_P L(f, P) = \inf_P \overline{U}(f, P)$$

Then $|L(f, P_3) - \overline{U}(f, P_3)| < \varepsilon$

By Prop. 1.12, $L(f, P_3) \le \overline{U}(f, P_3)$

$$0 \le \overline{U}(f, P_3) - L(f, P_3) < \varepsilon$$

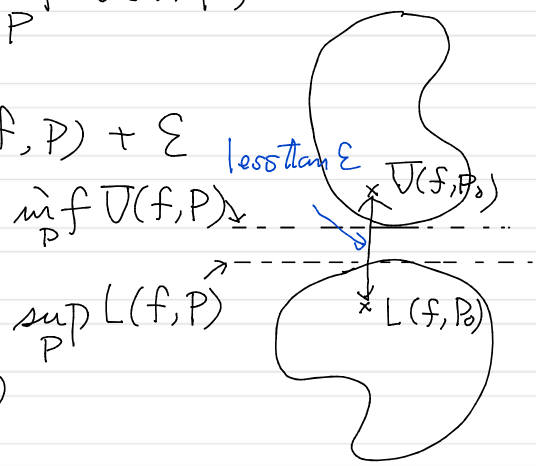
$(\Leftarrow)$

By Prop. 1.12, for any $\varepsilon > 0$, $\exists P_0 \in \Pi$ s.t.

$$\overline{U}(f, P_0) - L(f, P_0) < \varepsilon \quad , \quad \overline{U}(f, P_0) < L(f, P_0) + \varepsilon$$

Since $\sup_P L(f, P) \le \inf_P \overline{U}(f, P)$

$$\inf_P \overline{U}(f, P) \le \sup_P L(f, P) + \varepsilon$$



Since ε is any positive number,

$$\inf_P \overline{U}(f, P) = \sup_P L(f, P)$$

Thus, f is Riemann integrable. ///

<u>Definition 1.15</u> $\quad \operatorname{osc}_A f = \sup_{x \in A} f(x) - \inf_{x \in A} f(x)$

$P = \{I_1, I_2, \cdots, I_n\}$: partition.

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \sup_{I_k} f |I_k| - \sum_{k=1}^{n} \inf_{I_k} f |I_k|$$

$$= \sum_{k=1}^{n} \left(\sup_{I_k} f - \inf_{I_k} f \right) |I_k|$$

$$= \sum_{k} \operatorname{osc}_{I_k} f \, |I_k|$$

Proposition 1.16, $f, g : [a,b] \to \mathbb{R}$ bounded

g : integrable

$\exists C > 0$ s.t $\operatorname{osc}_I f \leq C \operatorname{osc}_I g \quad \forall I = [a', b'] \subset [a, b]$

Then

f is integrable.

(proof) $P = \{I_1, \cdots, I_n\}$

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \left(\sup_{I_k} f - \inf_{I_k} f \right) |I_k|$$

$$= \sum_{k=1}^{n} \operatorname{osc}_{I_k} f \, |I_k|$$

$$\leq \sum_{k=1}^{n} \operatorname{osc}_{I_k} g \, |I_k|$$

(Using Th 1.14) ///

Theorem 1.17 $f : [a,b] \to \mathbb{R}$ is integrable

$\iff$ $\exists \{P_n\}$: sequence of Partition

s.t. $\lim_{n \to \infty} |U(f, P_n) - L(f, P_n)| = 0$

///

$$d_n = \overline{U}(f, P_n) - L(f, P_n) \to 0$$