5/20 Integrability of continuous functions

Theorem 1.19 $f:[a,b]\to\mathbb{R}$ conti. $\Rightarrow$ f: Riemann integrable

(Proof) Our aime is to verify the condition of Th.1.14.

("Continuous functions are bounded" is well known.)

f is continuous at x_0

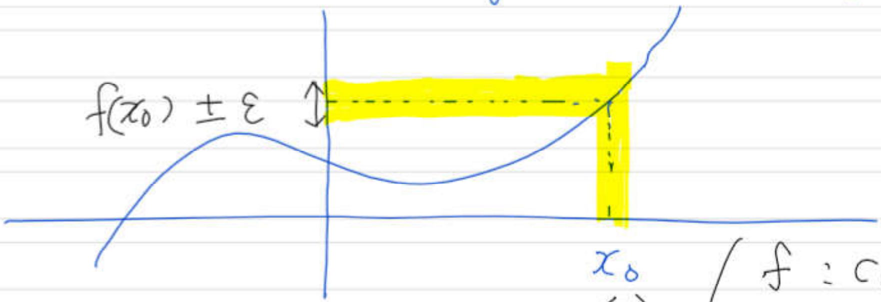
$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Even if we select very small error value ε (>0),
we can set allowance value $\delta > 0$ satisfying

$$|f(x) - f(x_0)| < \varepsilon \quad \text{if } |x - x_0| < \delta$$

The error is less than ε for all x in $(x_0 - \delta, x + \delta)$



(f : continoous
$\Leftrightarrow$ δ can be set for all $\varepsilon > 0$

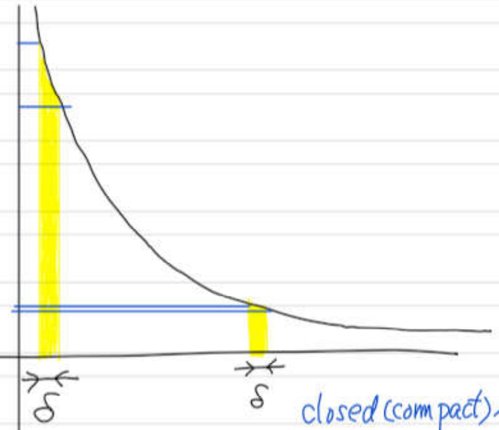
f : uniformly continuous

$$\Leftrightarrow \begin{pmatrix} \forall \varepsilon > 0 \ \exists \delta > 0 \ \text{s.t.} \\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \end{pmatrix}$$

For an arbitrary small error $\varepsilon > 0$,
a common allowance value δ can be selected which satisfies
$|f(x) - f(y)| < \varepsilon$ when $|x - y| < \delta$. δ does not depend on x and y.

($g(x) = \frac{1}{x}$ $((0,1]\to\mathbb{R})$ is continuous, however, it is not uniformly continuous.

For any fixed $\delta > 0$, the corresponding error values are getting larger as the points are getting nearer to 0.



closed (compact) interval

Fact : A continuous function defined on $[a, b]$ is uniformly continuous.

<< back to the proof of Th 1.19 >>

Using the uniform continuity of f, for any $\varepsilon > 0$, there exists $\delta > 0$ s.t.

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a} \quad \cdots ①$$

Let P_1, P_2 be partitions of $I = [a, b]$, we can construct a refinement $P' = \{I_k\}_{k=1}^{n}$ of P_1 and P_2 satisfying

$$\max_{k \leq n} |I_k| < \delta$$

Then by ① $\left| \sup_{x \in I_k} f(x) - \inf_{x \in I_k} f(x) \right| < \frac{\varepsilon}{b-a}$

$U(f, P_1) \geq U(f, P')$, $L(f, P_2) \leq L(f, P')$

We can select P_1, P_2

$$\inf_P U(f, P) + \varepsilon \geq U(f, P_1) \geq U(f, P')$$

$$\sup_P L(f, P) - \varepsilon \leq L(f, P_2) \leq L(f, P')$$

$$|U(f, P') - L(f, P')| = \sum_k \sup_{x \in I_k} f(x) |I_k|$$

$$- \sum_k \inf_{x \in I_k} f(x) |I_k|$$

$$= \sum_k \left| \sup_{x \in I_k} f(x) - \inf_{x \in I_k} f(x) \right| |I_k| \le \sum_k \frac{\varepsilon}{b-a} |I_k| = \frac{\varepsilon}{b-a} \sum |I_k|$$

$$= \varepsilon$$

$$\left| \inf_P \overline{U}(f,P) - \sup_P L(f,P) \right| \le \left(\overline{U}(f,P') - L(f,P') \right) + 2\varepsilon$$

$$\le 3\varepsilon$$

Thus, $\inf_P \overline{U}(f,P) - \sup_P L(f,P) = 0$ because $\varepsilon > 0$ is arbitrary small.

Therefore f is Riemann integrable. ///

Example $\quad f(x) = x^2 \qquad I = [0, 1]$

$P_n = \{ [\frac{k-1}{n}, \frac{k}{n}] \}_{k=1}^{n}$: partitions of I

$|P_n| = \max_k \left| \frac{k}{n} - \frac{k-1}{n} \right| = \frac{1}{n} \to 0$ as $n \to \infty$

$$I_{n,k} = \left[\frac{k-1}{n}, \frac{k}{n} \right], \quad P_n = \{ I_{n,k} \}$$

$$\alpha_k = \sup_{x \in I_{n,k}} f(x) = \frac{k^2}{n^2}, \quad \beta_k = \inf_{x \in I_{n,k}} f(x) = \frac{(k-1)^2}{n^2}$$

$$\alpha_k - \beta_k = \frac{k^2}{n^2} - \frac{(k-1)^2}{n^2} = \frac{(k+k-1)(k-(k-1))}{n^2}$$

$$= \frac{2k-1}{n^2}$$

$$\left(0 \le \frac{2k-1}{n^2} \le \frac{2n-1}{n^2} \le \frac{2n}{n^2} = \frac{2}{n} \xrightarrow{n \to \infty} 0 \right)$$