c

5/27 Properties of integral

Riemann integral has the following properties. (We will prove)

(1) Linearity.

$$\int_a^b c f(x)dx = c\int_a^b f(x)dx, \quad \int_a^b (f+g)dx = \int_a^b f dx + \int_a^b g dx$$

(2) Monotonicity. $f \le g$

$$\Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx$$

(3) Additivity $a < c < b$

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

For summation Σ, the following properties are natural.

(1)' $\sum_{k=1}^n c a_k = c\sum_{k=1}^n a_k, \quad \sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$

(2)' $a_k \le b_k \ (\forall k) \Rightarrow \sum_{k=1}^n a_k \le \sum_{k=1}^n b_k$

(3)' $\sum_{k=1}^m a_k + \sum_{k=m+1}^n a_k = \sum_{k=1}^n a_k$

1.6.1 Linearity

Theorem 1.23 $f:[a,b] \to \mathbb{R}$ integrable. $c \in \mathbb{R}$

$$\int_a^b cf(x)\,dx = c\left(\int_a^b f(x)\,dx\right)$$

(Proof) $\forall A \subset I$, $c > 0$

$$\sup_{x\in A} c f(x) = c \sup_{x\in A} f(x), \quad \inf_{x\in A} c f(x) = c \inf_{x\in A} f(x)$$

$$U(cf,P) = \sum_{k=1}^n \sup_{x\in I_k} cf(x)\,|I_k|, \quad P = \{I_k\}_{k=1}^n$$

$$= \sum_{k=1}^n c \sup_{x\in I_k} f(x)\,|I_k| = c\sum_{k=1}^n \sup_{x\in I_k} f(x)\,|I_k| = c\,U(f,P)$$

$$L(cf,P) = \sum_{k=1}^n \inf_{x\in I_k} cf(x)\,|I_k| = c\sum_{k=1}^n \inf_{x\in I_k} f(x)\,|I_k| = cL(f,P)$$

$$\inf_P U(cf,P) = \inf_P c\,U(f,P) = c\inf_P U(f,P)$$

$= c(\sup_P L(f,P)) = \sup_P L(cf,P))$

(if f is integrable)

$\Rightarrow$ cf is integrable, $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ //

In the case $c<0$ $\forall A \subset I$

$\sup_{x\in A} cf(x) = c \inf_{x\in A} f(x)$, $\inf_{x\in A} cf(x) = c \sup_{x\in A} f(x)$

$$U(cf, P) = \sum_k \sup(cf(x), I_k)|I_k|$$
$$= \sum_k c \inf(f(x), I_k)|I_k|$$
$$= c\sum_k \inf(f(x), I_k)|I_k| = cL(f,P)$$

$$L(cf, P) = \sum_k \inf(cf(x), I_k)|I_k|$$
$$= c\sum \sup_{x\in I_k} f(x)|I_k| = cU(f,P)$$

$$\inf_P U(cf,P) = \inf_P cL(f,P) = c\sup_P L(f,P)$$
$$= c\inf_P U(f,P) = \sup_P cU(f,P)$$
$$= \sup_C L(cf,P) \text{ (if } f \text{ is integrable)}$$

cf is integrable and $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ //

Theorem 1.24 f, g integrable $\Rightarrow$ $f+g$: integrable
and $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(proof)

For $\forall \varepsilon>0$, $\exists P_1, P_2, P_3 P_4 \in \pi$

$U(f, P_1) < \int_a^b f(x)dx + \varepsilon/2$ $L(f, P_3) > \int_a^b f(x)\,dx - \varepsilon/2$

$U(g, P_2) < \int_a^b g(x)dx + \varepsilon/2$ $L(g, P_4) > \int_a^b g(x)\,dx - \varepsilon/2$

$P = P_1 \cup P_2 \cup P_3 \cup P_4$ (Refinement)

$U(f, P) < \int_a^b f(x)dx + \varepsilon/2 \quad L(f, P) > \int_a^b f(x)\,dx - \varepsilon/2$

$U(g, P) < \int_a^b g(x)dx + \varepsilon/2 \quad L(g, P) > \int_a^b g(x)\,dx - \varepsilon/2$

$P = \{I_k\}_{k=1}^n$

$$\int_a^b f+g \; dx \le U(f+g) = \sum_{k=1}^n \sup_{x \in I_k} (f(x)+g(x)) \, |I_k|$$

$$\le \sum_{k=1}^n \left\{ \sup_{x\in I_k} f(x) + \sup_{x\in I_k} g(x) \right\} |I_k|$$

$$= \sum_{k=1}^n \sup_{x\in I_k} f(x) |I_k| + \sum_{k=1}^n \sup_{x\in I_k} g(x) |I_k|$$

$$< \int_a^b f(x)\,dx + \varepsilon/2 + \int_a^b g(x)\,dx + \varepsilon/2$$

$$= \left\{ \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \right\} + \varepsilon$$

$$\int_a^b f+g \; dx \ge L(f+g) = \sum_{k=1}^n \inf_{x\in I_k} (f(x)+g(x)) \, |I_k|$$

$$\ge \sum_{k=1}^n \left(\inf_{x\in I_k} f(x) + \inf_{x\in I_k} g(x) \right) |I_k|$$

$$= \sum_{k=1}^n \inf_{x\in I_k} f(x) |I_k| + \sum_{k=1}^n \inf_{x\in I_k} g(x) |I_k|$$

$$> \int_a^b f(x)\,dx - \frac{\varepsilon}{2} + \int_a^b g(x)\,dx - \frac{\varepsilon}{2}$$

$$= \left\{ \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \right\} - \varepsilon$$

Thus, we have

$$\int_a^b f(x)dx + \int_a^b g(x)\,dx - \varepsilon < \int_a^b f(x) + g(x)\,dx < \int_a^b f(x)dx + \int_a^b g(x)\,dx + \varepsilon,$$

and this implies

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)dx + \int_a^b g(x)\,dx$$

since ε is arbitrally small.

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Appendix

$$A < B + \varepsilon,\ \forall \varepsilon > 0 \Rightarrow A \leq B$$
$$A > B - \varepsilon,\ \forall \varepsilon > 0 \Rightarrow A \geq B$$

$\forall \varepsilon > 0$

$A_1 < B_1 + \varepsilon,\ A_2 < B_2 + \varepsilon,\ \cdots,\ A_n < B_n + \varepsilon$

$\Rightarrow A_1 + A_2 + \cdots + A_n \leq B_1 + B_2 + \cdots + B_n$