6/3 Further existence results

Proposition 1.35

$f, g : [a,b] \to \mathbb{R}$ $D = \{x \mid f(x) \neq g(x)\}$ is a finite set.

f, g : Riemann integrable

$$\Longrightarrow \int_a^b f(x)\,dx = \int_a^b g(x)\,dx$$

(Proof.) We prove the proposition for the case that $D = \{c\}$.

($f(c) \neq g(c)$ and $x \neq c \Rightarrow f(x) = g(x)$)

$\exists M > 0$ $\quad |f(x)|, |g(x)| \leq M \quad \forall x$

$$|f(c) - g(c)| \leq 2M, \quad |\sup_{x \in I'} f(x) - \sup_{x \in I} g(x)| \leq 2M$$

$\forall \varepsilon > 0$ (any positive number)

$$\delta = \frac{\varepsilon}{8M}$$

$P = \{I_k\}_{k=1}^n$ a partition. $|I_k| < \delta$

$$\left| \sum \sup_{x \in I_k} f(x) |I_k| - \sum \sup_{x \in I_k} g(x) |I_k| \right|$$

$$\leqq \sum \left| \sup_{x \in I_k} f(x) - \sup_{x \in I_k} g(x) \right| |I_k| \quad \cdots (1)$$

$\sup_{x \in I_k} f(x) \neq \sup_{x \in I_k} g(x)$ at most 2 intervals

$|\sup f(x) - \sup g(x)| \leq 2M$

$$(1) \leq 2 \times 2M \times \delta = \frac{\varepsilon}{2}$$

We can select P satisfying $U(f, P) < \int f\,dx + \frac{\varepsilon}{2}$

Then,

$$\int g(x)dx \leq U(g, P) \leq U(f, P) + \frac{\varepsilon}{2} \leq \int f\,dx + \varepsilon.$$

Similarly,

$$\int f(x)dx \leq \int g(x) + \varepsilon$$

Thus $\left|\int f(x)\,dx - \int g(x)\,dx\right| < \varepsilon$.

This implies $\int f\,dx = \int g\,dx$, since $\varepsilon > 0$ is an arbitrary small number.

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Example $f(x) = 0 \quad x \neq \frac{1}{2}, \quad f(\frac{1}{2}) = 1 \quad f : [0,1] \to \mathbb{R}$

$P_n = \{ [0, \frac{1}{2} - \frac{1}{n}], [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}], [\frac{1}{2} + \frac{1}{n}, 1] \}$.

$n \geq 3$

$$U(f, P_n) = \sup_{x \leq \frac{1}{2} - \frac{1}{n}} f(x) \left(\frac{1}{2} - \frac{1}{n}\right) + \sup_{x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}]} f(x)$$

$$+ \sup_{\frac{1}{2} + \frac{1}{n} \leq x} f(x) \left(\frac{1}{2} - \frac{1}{n}\right)$$

$$= 0 + 1 \times \frac{2}{n} + 0 = \frac{2}{n} \to 0$$

$L(f, P) \geq 0$ implies that $\int f(x)\, dx = 0$ ///

Proposition 1.39 $f : [a, b] \to \mathbb{R}$ bounded

f is integrable on $[a, r]$ for any $r \in [a, b)$

$\Rightarrow$ f is integrable on $[a\, b]$ and $\int_a^b f(x)\, dx = \lim_{r \nearrow b} \int_a^r f(x)\, dx$

(Proof) $\forall \varepsilon > 0, \quad \delta_0 \equiv \frac{\varepsilon}{4M} \wedge (b-a), \quad M = \sup |f(x)|$

$\exists P \in \Pi([a, r]) \quad (f : [a, r] \to \mathbb{R} \quad r = b - \delta_0)$

$$U(f, P) - L(f, P) < \frac{\varepsilon}{4}$$

$P' = P \cup \{[r, b]\} \in \Pi([a, b])$

$$|U(f, P) - U(f, P')| = |\sup_{x \in [r, b]} f(x)| \, |b - r| < M\delta_0 \leq \frac{\varepsilon}{4}$$

$$|L(f, P) - L(f, P')| = |\inf_{x \in [r, b]} f(x)| \, |b - r| < M\delta_0 \leq \frac{\varepsilon}{4}$$

$$|U(f, P') - L(f, P')| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon$$

f is integrable on $[a\, b]$ and

$$\int_a^b f(x)\, dx = \lim_{r \nearrow b} \int_a^r f(x)\, dx$$

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Example $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} \sin \frac{1}{x} & (0 < x < 1) \\ 0 & (x = 0) \end{cases}$$

$|f(x)| \leq 1$, (bounded) and

f is continuous on $[0, r]$ $\quad 0 \leq r < 1$

f is not continuous at $x = 0$

$\overline{\lim\limits_{x \to 0}} f(x) = 1$, $\underline{\lim\limits_{x \to 0}} f(x) = -1$, $|f(x)| \leq 1$

For any $\delta > 0$, $\exists\, x_1, x_2 \in [0, \delta]$ s.t.
$f(x_1) = 1$, $f(x_2) = -1$

f : continuous on $[0, r]$ $(r < 0)$, then
f is integrable $[0, r]$. Thus f satisfies the conditions of Th 1.39.
Therefore, f is integrable on $[a, b]$.