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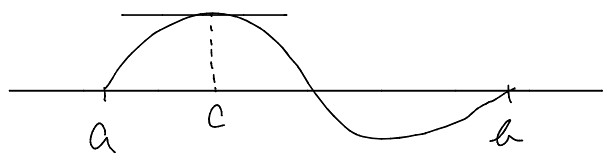
6/10 Fundamental Theorems 1

Preparations

◎ Roll's theorem : $f : [a, b] \to \mathbb{R}$ continuous.
differentiable on (a, b) , $f(a) = f(b)$
Then,

$$\exists c \in (a, b) \text{ s.t. } f'(c) = 0$$



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◎ Mean value theorem

f : continuous on $[a, b]$, differentiable on (a, b)
Then $\exists c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

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$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

satisfies the conditions of Roll's theorem.

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

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1.8.1 Fundamental theorem 1

F : continuous on $[a, b]$, differentiable on (a, b)
$f(x) = F'(x)$ is Riemman integrable.
Then $\int_a^b f(x)\,dx = F(b) - F(a)$
(Proof)
Using the integrability of f, for $\forall \varepsilon > 0$
$\exists P = \{t_0, t_1, \cdots, t_n\}$ s.t.

$|U(f,P) - L(f,P)| < \varepsilon \qquad L(f,P) \le \int_a^b f(x)\,dx \le U(f,P)$

$$F(b) - F(a) = F(t_n) - F(t_0)$$
$$= \sum_{k=1}^{n} (F(t_k) - F(t_{k-1})) \quad \cdots \text{①}$$

By the mean value theorem, we have: (for each $k = 1, \cdots, n$)

$\exists\, \hat{t}_k \in (t_{k-1}, t_k)$ s.t. $F(t_k) - F(t_{k-1}) = F'(\hat{t}_k)(t_k - t_{k-1}) = f(\hat{t}_k)(t_k - t_{k-1})$

$$\text{①} = \sum_{k=1}^{n} f(\hat{t}_k)(t_k - t_{k-1}) = \text{②}$$

$$\inf_{x \in [t_{k-1}, t_k]} f(x) \le f(\hat{t}_k) \le \sup_{x \in [t_{k-1}, t_k]} f(x)$$

$$\Rightarrow L(f,P) \le \text{②} \le U(f,P)$$

$$\Rightarrow \left| \text{②} - \int_a^b f(x)\,dx \right| < \varepsilon$$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$$

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Theorem 1.48

$f : [a, b] \to \mathbb{R}$ Rieman integrable on $[a, b]$,

continuous at a,

$$\lim_{h \downarrow 0} \frac{1}{h} \int_a^{a+h} f(x)\,dx = f(a)$$

(Proof) Fix an arbitrary $\varepsilon > 0$.

By the continuity at a, $\exists\, \delta > 0$

s.t. $|a - x| < \delta$ ($\Leftrightarrow x \in [a, a+\delta)$)

$$\Rightarrow |f(a) - f(x)| < \varepsilon$$

$0 < h < \delta$, $P = \{I_i\}_{i=1}^{n}$: partition of $[a, a+h]$

$I_i \subset [a, a+\varepsilon) \qquad |f(a) - f(x)| < \varepsilon \quad \forall x \in I_i$

$i = 1, \cdots, n$

$$\sup_{x \in I_i} f(x), \quad \inf_{x \in I_i} f(x) \in [f(a) - \varepsilon, f(a) + \varepsilon]$$

$$\left(f(a) - \varepsilon \le \inf_{x \in I_i} f(x) \le \sup_{x \in I_i} f(x) \le f(a) + \varepsilon \right)$$

$\forall i = 1, \cdots, n$

$$L(f, P) = \sum_{i=1}^{n} \inf_{x \in I_i} f(x) |I_i|$$

$$U(f, P) = \sum_{i=1}^{n} \sup_{x \in I_i} f(x) |I_i|$$

$$\sum_{i=1}^{n} |I_i| = h$$

$$h(f(a) - \varepsilon) = \sum_{i=1}^{n} (f(a) - \varepsilon)|I_i| \le \sum_{i=1}^{n} \inf_{x \in I_i} f(x) |I_i|$$

$$\sum_{i=1}^{n} \sup_{x \in I_i} f(x) \le \sum_{i=1}^{n} (f(a) + \varepsilon)|I_i| = h(f(a) + \varepsilon)$$

$$f(a) - \varepsilon \le \frac{1}{h} L(f, P) \le \frac{1}{h} U(f, P) \le f(a) + \varepsilon$$

$$f(a) - \varepsilon \le \frac{1}{h} \sup_{P} L(f, P) \le \frac{1}{h} \inf_{P} U(f, P) \le f(a) + \varepsilon$$

By the integrability

$$\frac{1}{h} \sup_{P} L(f, P) = \frac{1}{h} \inf_{P} U(f, P) = \frac{1}{h} \int_a^{a+h} f(x)\, dx$$

then $\quad f(a) - \varepsilon \le \frac{1}{h} \int_a^{a+h} f(x)\, dx \le f(a) + \varepsilon$

This implies that

$$\lim_{h \searrow 0} \frac{1}{h} \int_a^{a+h} f(x)\, dx = f(a)$$

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Similarry we can show the following properties.

f is continuous at b

$$\Rightarrow \lim_{h \searrow 0} \frac{1}{h} \int_{b-h}^{b} f(x)\,dx = f(b)$$

f is continuous at $c \in (a, b)$,

$$\Rightarrow \lim_{h \searrow 0} \frac{1}{2h} \int_{c-h}^{c+h} f(x)\,dx = f(c)$$

f is continuous on $[a, b]$

$$F(x) = \int_a^x f(t)\,dt$$

$$F'(c) = f(c) \qquad \forall c \in (a, b)$$