# 6/17 Consequence of the fundamental theorems

**Example 1.52** $\frac{d}{dx}\left[\frac{1}{p+1}x^{p+1}\right] = x^p$ implies

$$\int_0^1 x^p\,dx = \frac{1}{p+1} \quad (\text{Using TH } 1.45)$$

$p > 0 \quad f(x) = x^p \quad (x > 0)$

$P_n = \left\{\left[\frac{j-1}{n}, \frac{j}{n}\right]\right\}_{n=1}^{n}$ : partitions of $I$ $(n = 1, 2, \cdots)$

$$U(f, P_n) = \sum_{k=1}^{n} \sup_{x \in \left[\frac{j-1}{n}, \frac{j}{n}\right]} f(x)\frac{1}{n}$$

$$= \sum_{k=1}^{n}\left(\frac{j}{n}\right)^p \frac{1}{n} = \frac{1}{n^{p+1}}\sum_{j=1}^{n} j^p$$

$$\longrightarrow \int_0^1 x^p\,dx = \left(\frac{x^{p+1}}{p+1}\right)_0^1 = \frac{1}{p+1}$$

**Theorem 1.53** $f, g : [a, b] \to \mathbb{R}$ $f', g'$ : integrable.

$$\Rightarrow \int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx$$

(Proof)

$$(fg)' = fg' + f'g$$

By TH 1.26, TH 1.45

$$\int_a^b fg'\,dx + \int_a^b f'g\,dx = \int_a^b (fg)'\,dx$$

$$= f(b)g(b) - f(a)g(a)$$

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**Example 1.54** $n = 0, 1, 2, \cdots$

$$I_n(x) = \int_0^x t^n e^{-t}\,dt$$

If $n \geq 1$, $f(t) = t^n$ $g'(t) = e^{-t}$ $(g(t) = -e^{-t})$

Using TH 1.53 we have:

$$I_n(x) = -x^n e^{-x} + n\int_0^x t^{n-1}e^{-t}\,dt = -x^n e^{-x} + nI_{n-1}(x)$$

$$I_0(x) = \int_0^x e^{-t}\,dt = 1 - e^{-x} \quad \cdots ①$$

$$I_n(x) = n!\left[1 - e^{-x}\sum_{k=0}^{n}\frac{x^k}{k!}\right] \quad ② \quad (0! = 1 : \text{definition.})$$

$\because$ ① implies the case $n = 0$.

Asume ② $k \leq n-1$,

$$\begin{aligned} I_n(x) &= -x^n e^{-x} + n I_{n-1}(x) \\ &= -x^n e^{-x} + n(n-1)!\left[1 - e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{k!}\right] \\ &= n!\left[1 - e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{k!} - \frac{x^n}{n!}\right] \\ &= n!\left[1 - e^{-x}\sum_{k=0}^{n}\frac{x^k}{k!}\right] \end{aligned}$$

By the mathematical induction, we have ② for any $n \in \mathbb{N}$.

$$\lim_{x\to\infty} e^{-x}\frac{x^k}{k!} = 0 \quad \text{for any } k \in \mathbb{N}$$

Then
$$\begin{aligned} \lim_{x\to\infty} I_n(x) &= n!\left(1 - \lim_{x\to\infty}\sum\frac{x^k}{k!}e^{-x}\right] \\ &= n! \end{aligned}$$

$$\int_0^\infty t^n e^{-t}\,dt = \lim_{r\to\infty}\int_0^r t^n e^{-t}\,dt = n!$$

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$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ ($\Gamma$ - function)

$\Gamma(n) = (n-1)!$

Theorem 1.55 (Change of variable)

$g : I = [a, b] \to \mathbb{R}$ $g'$ : integrable on $I$. $f : [g(a), g(b)]$ continuous

$$\Rightarrow \int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \quad (g(a) < g(b))$$

(proof) We only consider the case $g' > 0$.

$F(x) = \int_a^x f(u)\,du$

$(F\circ g)'(x) = f(g(x))g'(x)$ By TH 1.45.

$$\begin{aligned} \int_a^b f(g(x))g'(x)\,dx &= \int_a^b (F\circ g)'(x)\,dx \\ &= F(g(b)) - F(g(a)) \end{aligned}$$

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Example 1.56 $\quad g(x) = x^3$

$I = [-a, a]$ , $J = [-a^3, a^3]$ $\quad f$: continuous on $J$.

$$\int_{-a}^{a} f(x^3)\, 3x^2 dx = \int_{-a^3}^{a^3} f(u)\, du$$

$$\int_{\alpha}^{\beta} f(x^2)\, 2x\, dx = \int_{\alpha^2}^{\beta^2} f(t)\, dt .$$

$\alpha = -a$, $\beta = a$ $\quad (a > 0) \Rightarrow \int_{a^2}^{a^2} f(t)dt = 0$

$P_n = \left\{ \left[\frac{k-1}{n}, \frac{k}{n}\right] \right\}_{k=-n+1}^{n} \qquad I_k^{(n)} = \left[\frac{k-1}{n}, \frac{k}{n}\right]$

$k = -n+1, \cdots, n$ .

$k \geq 1 \Rightarrow I_k^{(n)} = [\alpha, \beta] \Rightarrow I_{-(k-1)}^{(n)} = [-\beta, -\alpha]$

For large enough $n$,

$$\begin{pmatrix} \sup_{x \in I_k^{(n)}} 2x f(x^2) \\ \inf_{x \in I_k^{(n)}} 2x f(x^2) \end{pmatrix} \fallingdotseq \begin{pmatrix} -\sup_{x \in I_{-(k-1)}^{(n)}} 2x f(x^2) \\ -\inf_{x \in I_{-(k-1)}^{(n)}} 2x f(x^2) \end{pmatrix}$$

$$\Rightarrow U(2x f(x^2), P_n) \fallingdotseq 0$$

$$L(2x f(x^2), P_n) \fallingdotseq 0$$

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