6/24 Examples

Example 1.57 $f(x) = \int_1^x \frac{1}{t}\,dt \quad x \in [1, \infty)$

f is another definition for $\log x$.

$x, y \in [1, \infty) \quad f(x \times y) = f(x) + f(y)$

$$\int_1^y \frac{1}{t}\,dt = \int_x^{xy} \frac{1}{t}\,dt \quad \forall x > 0$$

By the definition for Riemann integral, for any $\varepsilon > 0$

$\exists P = \{t_0, t_1, \cdots, t_n\}$ (a partition) s.t. $(1 = t_0 < \cdots t_n = y)$

$|U(g, P) - L(g, P)| < \varepsilon$, ($g(t) = \frac{1}{t}$, $L(g,P) \le \int_0^y g(t)\,dt \le U(g,P)$)

$\overline{g}_j = \sup_{t_{j-1} \le t \le t_j} \frac{1}{t}$, $\underline{g}_i = \inf_{t_{j-1} \le t \le t_j} \frac{1}{t}$

$$\left(\overline{g}_j = \frac{1}{t_{j-1}}, \quad \underline{g}_j = \frac{1}{t_j} \right)$$

$U(g, P) = \sum_{j=1}^n \overline{g}_j (t_j - t_{j-1})$, $L(g, P) = \sum_{j=1}^n \underline{g}_j (t_j - t_{j-1})$

$$U(g, P) = \sum_{j=1}^n \overline{g}_j \times \frac{1}{x} (x t_j - x t_{j-1})$$

$$= \sum_{j=1}^n \frac{1}{x t_{j-1}} (x t_j - x t_{j-1})$$

$$L(g, P) = \sum_{j=1}^n \frac{1}{x t_j} (x t_j - x t_{j-1})$$

$P' = \{x t_0, x t_1 \cdots x t_n\}$ is a partion of $[x, xy]$

$\frac{1}{x t_{j-1}} = \sup_{t \in [x t_{j-1}, x t_j]} \frac{1}{t}$, $\frac{1}{x t_j} = \inf_{t \in [x t_{j-1}, x t_j]} \frac{1}{t}$

$U(g, P) = U(g, P')$, $L(g, P) = L(g, P')$ $\cdots$ ①

$|U(g, P') - L(g, P')| < \varepsilon$, ($L(g, P') \le \int_x^{xy} g(t)\,dt \le U(g, P')$)

By ① we have

$$\int_1^y g(t)\,dt = \int_x^{xy} g(t)\,dt \quad \cdots ②$$

$$f(x) + f(y) = \int_1^x g(t)\,dt + \int_1^y g(t)\,dt$$

$$= \int_1^x g(t)\,dt + \int_x^{xy} g(t)\,dt \quad \text{using ②}$$

$$= \int_1^{xy} g(t)\,dt = f(xy)$$

Example 1.58

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$E = \int_{-\infty}^{\infty} e^{-t^2} dt$ is calculated as follows.

$$E^2 = \int_{-\infty}^{\infty} e^{-t^2} dt \int_{-\infty}^{\infty} e^{-s^2} ds$$

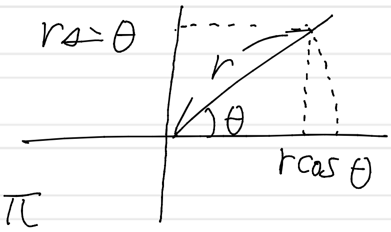
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(t^2+s^2)} dt\, ds \quad \cdots ①$$

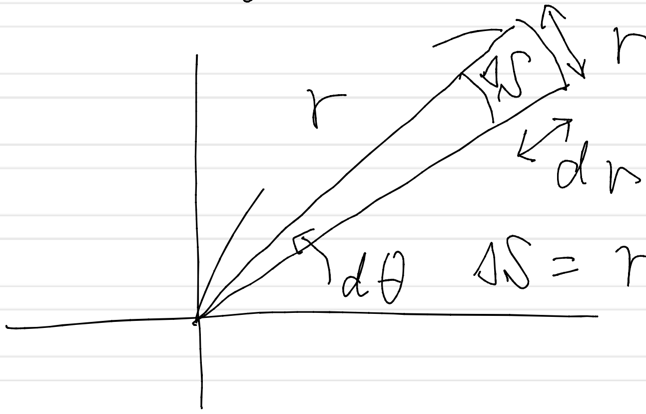
$t = r\cos\theta,\ s = r\sin\theta,\ dt\,ds = r\,dr\,d\theta$

$\{(t,s) \mid t, s \in \mathbb{R}\} = \{(r\cos\theta, r\sin\theta) \mid \theta \in [0, 2\pi],\ r \in [0, \infty)\}$

$$① = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{2} e^{-r^2}\right]_0^{\infty} d\theta = \pi$$





$\Delta S = r\,dr\,d\theta$

$$\mathrm{erf}(x) = \sum_{k=0}^{\infty} a_k x_k \qquad \text{Taylar decomposition.}$$

$$a_k = \frac{1}{k!} \mathrm{erf}^{(k)}(0)$$

(It is well known that this is good approximation)

We also have:

$$\mathrm{erf}'(x) = \frac{d}{dx}\left(\frac{2}{\pi} \int_0^x e^{-t^2} dt\right)$$

$$= \frac{2}{\pi} e^{-t^2}$$

$(e^x)' = e^x$, $(e^{x^2})' = 2xe^{x^2}$ . . . .

Then a_k's are easily and concletely calculated using these properties.

Thus the conclete values of the function erf$(\cdot)$ can be calculated using a computer. very rapidly and very correctly.

And a remarkable point of this calculation is that integral values can be approximated using differential coefficients.