7/01 Integrals and sequences of functions

$f_n \to f \quad \Rightarrow \quad \int f_n\,dx \to \int f\,dx$ (?)

$f_n \to f$ pointwise on A

$\Leftrightarrow \quad f(x) = \lim_{n\to\infty} f(x)$ for $\forall x \in A$

$f_n \to f$ uniformly on A

$\Leftrightarrow \forall \varepsilon > 0, \exists N$ s.t.

$n \geq n \Rightarrow |f_n(x) - f(x)| < \varepsilon$ for any $x \in A$

$\Leftrightarrow \quad \sup_{x\in A} |f_n(x) - f(x)| \to 0$ as $n \to \infty$

Theorem 1.61 $\quad f_n \to f$ uniformly on $[a,b]$ $(n\to\infty)$.

f_n : integrable $\forall n \in \mathbb{N}$

$\Rightarrow f$: integrable and $\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$

(proof)

$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}$ s.t.

$n \geq N \Rightarrow |f(x) - f_n(x)| < \frac{\varepsilon}{2(b-a)} \quad \forall x \in [a,b]$

$\Rightarrow f_n(x) - \frac{\varepsilon}{2(b-a)} < f(x) < f_n(x) + \frac{\varepsilon}{2(b-a)}$

By Prop. 1.30. $\quad x \in [a,b]$

$$\int_a^b \left(f_n - \frac{\varepsilon}{2(b-a)}\right) dx \leq L(f) \leq U(f) \leq \int_a^b \left(f_n + \frac{\varepsilon}{2(b-a)}\right) dx$$

$$\int_a^b f_n(x)\,dx - \frac{\varepsilon}{2} \leq L(f) \leq U(f) \leq \int_a^b f_n(x)\,dx + \frac{\varepsilon}{2}$$

$$|U(f) - L(f)| \leq \varepsilon \Rightarrow U(f) = L(f)$$

$\Rightarrow f$ is integrable

$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx \quad (n\to\infty)$$

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \leq \int_a^b |f_n(x) - f(x)|\,dx$$

$$\leq \sup_{x\in[a,b]} |f_n(x) - f(x)| \int_a^b dx \to 0$$

//

Example 1.62 $\quad f_n(x) = \dfrac{n + \cos x}{n e^{x} + \sin x} \to e^{-x}$

(converges uniformly)

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 e^{x} dx = e - 1$$

$$\sup_{x\in[0,1]} \left| \frac{n + \cos x}{n e^{x} + \sin x} - e^{-x} \right|$$

$$= \sup_{x\in[0,1]} \frac{|n + \cos x - n - e^{-x}\sin x|}{|n e^{x} \sin x|}$$

$$\leqq \sup_{x\in[0,1]} \frac{1}{n} \frac{|\cos x| + |e^{-x}\sin x|}{|e^{x}\sin x|}$$

Example 1.63 $\quad f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$R > 0$: radius of convergence.

$R = \sup\{|x| : \sum_{j=0}^{\infty} a_j x^j \text{ converges}\}$

$$\left(\begin{array}{l} \sum_{j=0}^{\infty} a_j x^j \text{ converges uniformly on } \{x \mid |x| \leq r\} \\ r < R \end{array} \right.$$

$$\int_0^x f(t)\,dt = a_0 x + \frac{a_1}{2} x^2 + \cdots \quad \left(\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} \right)$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \quad \left(= \sum_{k=0}^{\infty} x^k \right) \ |x| < 1$$

integral

$$\log\left(\frac{1}{1-x}\right) = x + \frac{x^2}{2} + \cdots \quad \left(= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \right) \ |x| < 1$$

$$\log 2 = \log\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\frac{1}{2}\right)^{k+1}$$

Theorem 1.64 $f_n : (a,b) \to \mathbb{R}$

f'_n : integrable on (a,b). $f_n(x) \to f(x)$ $(\forall x \in (a,b))$

$f'_n \to g$ uniformly on (a,b) as $n \to \infty$.

g : continuous on (a,b)

$\Rightarrow$ f : differentiable on (a,b), $f'(x) = g$

(Proof)

Choose $c \in (a,b)$, $(a < c < b)$.

$$f_n(x) = f_n(c) + \int_c^x f'_n(x)\,dx \qquad x \in (a,b)$$

$$\left(f_n(c) - \int_x^c f'_n(x)\,dx \right)$$

$f_n(x) \to f(x)$ $(\forall x)$, $f'_n(x) \to g(x)$ uniformly

$$\Rightarrow \quad f(x) = f(c) + \int_c^x g(t)\,dt$$

$$\Rightarrow \quad f'(x) = g'(x) \qquad (\text{By Th } 1.50)$$

///

This theorem shows that

$\{f'_n(x)\}$ converges uniformly on (a,b)

, $f'_n(x)$ are continuous on (a,b),

and $f_n(x) \to f(x)$ $(\forall x)$

$\Rightarrow$ $f'(x)$ continuous.

///