7/8 Improper integrals.

Def 1.67 $f\,(a, b] \to \mathbb{R}$, integrable on $(c, b]$ $\forall c \in (a, b]$,

$$\Rightarrow \int_a^b f(x)\,dx \equiv \lim_{c \searrow a} \int_c^b f(x)\,dx$$

in the case the above limit exists.

$f : [a, b) \to \mathbb{R}$, integrable on $[a, c)$ $\forall c \in [a, b)$

$$\Rightarrow \int_a^b f(x)\,dx \equiv \lim_{c \nearrow b} \int_a^c f(x)\,dx$$

in the case the above limit exists.

Def 1.69 $f : [a, \infty) \to \mathbb{R}$, integrable on $[a, b]$, $\forall b > a$.

$$\Rightarrow \int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

in the case the avobe integral exists.

Example 1.68, 70 $P > 0$, $P \neq 1$

$$\int_1^\infty \frac{1}{x^P}\,dx = \lim_{r \to \infty} \int_1^r \frac{1}{x^P}\,dx = \lim_{r \to \infty} \left[\frac{1}{1-P} \frac{1}{x^{P-1}}\right]_1^r$$

$$= \lim_{r \to \infty} \frac{1}{P-1}\left(\frac{1}{r^{P-1}} - 1\right) = \begin{cases} \frac{1}{P-1} & P > 1 \\ \infty & P \geqq 0 \end{cases}$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{r \to \infty} \int_1^r \frac{1}{x}\,dx$$

$P \neq 1$

$$= \lim_{r \to \infty} [\log x]_1^r = \lim_{r \to \infty} \log r = \infty$$

$$\int_0^1 \frac{1}{x^P}\,dx = \lim_{r \searrow 0} \int_r^1 \frac{1}{x^P}\,dx$$

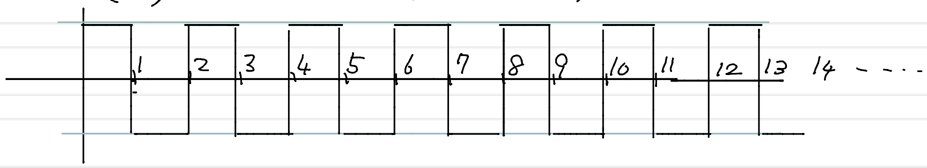
$$= \lim_{r \searrow 0} \frac{1}{P-1}\left(1 - \frac{1}{r^{P-1}}\right) = \begin{cases} \frac{1}{1-P}, & P < 1 \\ \infty & P > 1 \end{cases}$$

$$\int_0^1 \frac{1}{x}\,dx = \lim_{r \searrow 0} [\log x]_r^1$$

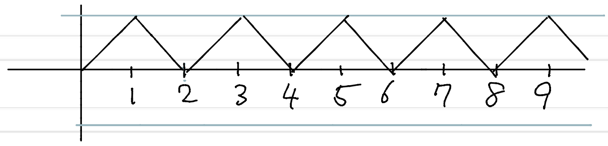
$$= \lim_{r \searrow 0} (-\log r) = \infty$$

Example 1.71 $f:[0,\infty)\to\mathbb{R}$

$$f(x)=(-1)^n \quad x\in[n,n+1] \;, \quad n=0,1,2\cdots$$



$$\int_0^a f(x)\,dx = \begin{cases} a-n & x\in[n,n+1] \quad n:\text{even} \\ 1-(a-n) & x\in[n,n+1] \quad n:\text{odd} \end{cases}$$



$F(a)=\int_0^a f(x)\,dx$ does not converge as $a\to\infty$.

Eample 1.72 $f:[0,1]\to\mathbb{R}$ continuous. $0<C<1$

$$\int_0^1 \frac{1}{\sqrt{|x-C|}} f(x)\,dx = \lim_{\varepsilon\searrow 0}\int_0^{C-\varepsilon}\frac{1}{\sqrt{C-x}}f(x)\,dx + \lim_{\varepsilon\searrow 0}\int_{C+\varepsilon}^1 \frac{1}{\sqrt{x-C}}f(x)\,dx$$

(These limits exist and we can define the above integral.)

$$\begin{cases} f(x)\dfrac{1}{\sqrt{C-x}} : \text{continuous. on } [0, C-\varepsilon] \quad (\forall\varepsilon>0) \text{ (small enough)} \\ f(x)\dfrac{1}{\sqrt{x-C}} : \text{continuous. on } [C+\varepsilon, 1] \quad (\forall\varepsilon>0) \end{cases}$$

Consider a simple case that f is continuos on $[0,1]$

Then $\frac{f(x)}{\sqrt{x}}$ is inlegable on $(0,1]$.

$$\rho(x) \equiv \int_x^1 \frac{1}{\sqrt{t}} f(t)\,dt$$

$$M = \sup_{x\in[0,1]} |f(x)| < \infty$$

(In general a continuous function on $[0,1]$ is bounded)

For any $\varepsilon > 0$, set $\delta = \frac{\varepsilon^2}{4M^2}$,

$x_1, x_2 \in (0, \delta)$ (Without loss of generality, $x_1 \leq x_2$)

$$\Rightarrow |P(x_2) - P(x_1)| = \left| \int_{x_1}^{x_2} \frac{1}{\sqrt{t}} f(t)\, dt \right|$$

$$\leq \int_{x_1}^{x_2} \frac{1}{\sqrt{t}} |f(t)| dt$$

$$\leq \int_{x_1}^{x_2} \frac{M}{\sqrt{t}} dt = 2M(\sqrt{x_2} - \sqrt{x_1})$$

$$\leq 2M\sqrt{x_2} < 2M\sqrt{\delta} = 2M\sqrt{\frac{\varepsilon^2}{4M^2}}$$

$$= \varepsilon$$

Therefore $P(x) \to \exists \int_0^1 \frac{1}{\sqrt{t}} f(t)\, dt$.

Returning to the setting of the example, we can prove the integrabilities of the functions:

$$f(x) \frac{1}{\sqrt{c - x}} : \quad \text{on } [a, c)$$

$$f(x) \frac{1}{\sqrt{x - c}} \cdot \quad \text{on } (c, b]$$

similally with the above arguments.