For sum of real numbers.

$$\sum_{k=1}^{\infty}|a_k|<\infty \Rightarrow \sum_{k=1}^{\infty}a_k \text{ converges.}$$

$$\left(\Leftrightarrow \exists \lim_{n\to\infty}\sum_{k=1}^{n}a_k \overset{=\alpha}{} \text{ exist} \left(\forall\varepsilon>0, \exists N\in\mathbb{N},\ n\geq N \Rightarrow \left|\sum_{k=1}^{n}a_k-\alpha\right|<\varepsilon\right)\right)$$

Example

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

k : even $\Rightarrow$ $k=2m$, k : odd $\Rightarrow$ $k=2m-1$,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = \sum_{m=1}^{\infty}\frac{(-1)^{2m-1+1}}{2m-1}+\frac{(-1)^{2m+1}}{2m} \quad \leftarrow \left(\frac{1}{2m-1}-\frac{1}{2m}\right)\geq 0$$

$$= \sum_{m=1}^{\infty}\frac{1}{2m-1}-\frac{1}{2m}$$

$$= \sum_{m=1}^{\infty}\frac{1}{2m(2m-1)} = \sum_{m=1}^{\infty}\frac{1}{4m^2-2m}$$

$$4m^2-2m-3m^2 = m^2-2m+1-1$$

$$=(m-1)^2-1 \geq 0 \quad m\geq 2$$

$$\sum_{m=1}^{\infty}\frac{1}{4m^2-2m} = \frac{1}{2}+\sum_{m=2}^{\infty}\frac{1}{4m^2-2m}$$

$$\leq \frac{1}{2}+\sum_{m=2}^{\infty}\frac{1}{3m^2}$$

$$\sum_{m=2}^{\infty}\frac{1}{3m^2} = \sum_{m=2}^{\infty}\int_{m-1}^{m}\frac{1}{3m^2}\,dx$$

$$\leq \frac{1}{3}\sum_{m=2}^{\infty}\int_{m-1}^{m}\frac{1}{x^2}\,dx$$

$$= \frac{1}{3}\int_{1}^{\infty}\frac{1}{x^2}\,dx$$

$$= \frac{1}{3}\left[-\frac{1}{x}\right]_{1}^{\infty} = \frac{1}{3}<\infty$$

This implies that $\displaystyle\sum_{m=1}^{\ell}\frac{1}{4m^2-2m}$: converges. $(\ell\to\infty)$

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\begin{cases} = \displaystyle\sum_{m=1}^{\ell}\frac{1}{4m^2-2m} & n=2\ell \text{ (even)} \\ = \displaystyle\sum_{m=1}^{\ell}\frac{1}{4m^2-2m}-\frac{1}{2\ell+1} & n=2\ell+1 \text{ (odd)} \end{cases}$$

$\frac{1}{2l+1} \to 0 \quad (\text{as } k, l \to \infty)$

Therefore, $\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$ converges.

On the other hand,

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$$
$$= \sum_{k=1}^{\infty} \int_k^{k+1} \frac{1}{k} dx$$
$$\geq \sum_{k=1}^{\infty} \int_k^{k+1} \frac{1}{x} dx$$
$$= \int_1^{\infty} \frac{1}{x} dx = \lim_{p \to \infty} [\log x]_1^p = \infty$$

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges conditionally

does not converge absolutely

Definition 1.74

$\int_a^b f(x)\,dx$: converges absolutely

$\Leftrightarrow \quad \int_a^b |f(x)|\,dx < \infty$

$\int_a^b f(x)\,dx$: converges conditionally

$\Leftrightarrow \quad \int_a^b |f(x)|\,dx = \infty, \ \int_a^b f(x)\,dx$ converges.

Theorem 1.75 $\quad f, g : I \to \mathbb{R}, \ |f(x)| \leq g(x) \quad x \in I$

$\int_I g(x)\,dx < \infty$ (I : finite or infinite interval.)

$\Rightarrow$

$\int_I f(x)\,dx$ converges absolutely.

(Proof) $\quad I = [a, \infty)$

We consider the case that

f, g are integrable on $[a, r]$ for an $r > a$.

Since $M = \int_a^\infty g(x)dx < \infty$, for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$y_1, y_2 \geq N \ (y_1 < y_2) \Rightarrow \int_{y_1}^{y_2} g(x)dx < \varepsilon$

$$\left| \int_a^{y_2} f(x)dx - \int_a^{y_1} f(x)dx \right|$$

$$= \left| \int_{y_1}^{y_2} f(x)dx \right| \leq \int_{y_1}^{y_2} |f(x)|dx \leq \int_{y_1}^{y_2} g(x)dx < \varepsilon$$

Similarlly we have.

$$\int_a^{y_2} |f(x)|dx - \int_a^{y_1} |f(x)|dx \leq \int_{y_1}^{y_2} g(x)dx < \varepsilon.$$

Therefore $\int_a^\infty f(x)dx$ congerges absolutely.

When $I = (a, b]$ and $\int_r^b g(x)dx \leq {}^\exists M < \infty$, we have $\forall r > a$

(M does not depend on $r > a$.)

For any $\varepsilon > 0$ $\exists r_0 > 0$ s.t. $\int_a^{r_0} g(x)dx < \varepsilon$

$r_1, r_2 < r_0 \quad (r_1 < r_2)$

$$\Rightarrow \left| \int_{r_1}^b f(x)dx - \int_{r_2}^b f(x)dx \right| = \left| \int_{r_1}^{r_2} f(x)dx \right|$$

$$\leq \int_{r_1}^{r_2} |f(x)| \cdot dx \leq \int_{r_1}^{r_2} g(x)dx < \varepsilon$$

Therefore $\int_a^b f(x)dx$ converges absolutely.

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Example $\int_0^\infty e^{-x^2}dx$ converges absolutely.

$$\int_0^\infty e^{-x^2}dx = \sum_{k=1}^\infty \int_{k-1}^k e^{-x^2}dx$$

$$\leq \sum_{k=1}^\infty \int_{k-1}^k e^{-(k-1)^2}dx = \sum_{k=1}^\infty e^{-(k-1)^2}$$

$$\leq \sum_{l=0}^\infty e^{-l} = \lim_{N\to\infty} \frac{1-e^{-N}}{1-e^{-1}} = \frac{1}{1-e} < \infty$$

$$\int_1^\infty \frac{1}{x^p}dx \begin{cases} < \infty & p > 1 \\ = \infty & p \leq 1 \end{cases}$$

$$\int_0^1 \frac{1}{x^p}dx \begin{cases} = \infty & p \geq 1 \\ < \infty & p < 1 \end{cases}$$