On Two Generalizations for k-additivity

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Abstract. There are two generalizations for k-additive set functions: constructive k-additivity and formulaic k-additivity. We study some properties around these concepts and their relations. A constructively k-additive set function is always formulaic k-additive. For a distorted measure, these two concepts are equivalent. Under certain conditions of "bounded variation" and "continuity at the $\emptyset$," we prove the constructive k-additivity for a formulaic k-additive set function.

Keywords: fuzzy measure· monotone measure· k-order additivity· Möbius transform · distorted measure

1 Introduction

The concept of k-order additivity was originally introduced for a monotone measure on a finite set using the Möbius transform (see for example [1],[2], [3]). This is an important concept to reduce the complexity of non-additive measures. As an example for the identification of a non-additive measure, we need $2^n - 1$ parameters to express a general set function defined on a set with cardinality n; however, only $(n^2 + n)/2$ parameters are required for a two-additive measure. Assuming that one uses linear regression for the identification of a non-additive measure, we use a covariance matrix. For the covariance matrix for a 10 elements' set, the number of elements can be reduced to 3025 from 1048576.

There are two approaches to define the k-additivity for non-discrete set functions. One is a constructive approach and the other uses a formulaic relation among the terms of the Möbius transform. R. Mesiar gave the first generalization for a k-additive non-additive measure [4] through the constructive approach. Fukuda, Honda, and Okazaki [5] proved a monotone decreasing convergence theorem for the Pan Integral with respect to a monotone measure of this type. Such monotone measures can be described by several σ-additive signed measures defined on set spaces, the precise definition of which will be given in section 2, and this is suitable to estimate integral values. Honda, Fukuda, and Okazaki [6] gave the definition of formulaic k-additivity. The Möbius transform was extended to non-discrete set functions for this definition, and the definition and some equivalent conditions were described using this Möbius transform. A distorted measure μ is a set function , which can be represented by $\mu(A) = f(m(A))$ using a probability measure $m(\cdot)$ and a non decreasing function f. The formulaic k-additivity is essentially equivalent to the fact that the distortion function f is a k-order polynomial ([6].)

Briefly, constructive k-additivity is useful for estimations or calculations, and formulaic k-additivity is a natural extension of k-additivity for the finite element case. The present study attempts to show that these two definitions are essentially the same. This problem is naturally valid on a finite set or on a finite σ-algebra. We analyze the structures of our settings in the finite case and describe the relations between the two generalizations for k-additivity. There are some natural conditions for this extension. A σ-additive measure satisfies continuity from above and below, and this is a key property for the extension of measures. A non-additive set function is not always continuous from above or below, but we assume these continuities.

Under these settings, we show the following properties in this paper.

(a) All constructive k-additive set functions are formulaic k-additive.
(b) A distorted measure is formulaic k-additive if and only if it is constructively k-additive.
(c) Consider a set function μ defined on a countably generated σ-algebra. Then, if a formulaic k-additive set function μ satisfies certain conditions of "bounded variation" and "fine continuity at $\emptyset$," μ is constructively k-additive.

By showing the above problems, we will try to make sure the richness of the concepts of k-additivity.

2 Definitions and notations

Throughout this paper, $(X, \mathcal{B})$ denotes a measurable space, that is, X is a non-discrete set and $\mathcal{B}$ is a σ-algebra ($\mathcal{B} \subset 2^X$). We assume that any set function μ defined on $\mathcal{B}$ satisfies $\mu(\emptyset) = 0$. We also assume that all one-point sets are measurable. Let n be a positive integer. We define an n-set space $X^{(n)}$ as follows.

$$X^{(n)} = \{\{x_k\}_{k=1}^n \subset X,\ x_j \neq x_k \text{ if } j \neq k\}.$$

$X^{(n)}$ can be also represented by

$$X^{(n)} = \{(x_1, \ldots, x_n) \in X^n : x_j \neq x_k \text{ if } j \neq k\}/\sim,$$

where the equivalence relation $\sim$ is defined by:

$$\begin{aligned} & (x_1, \ldots, x_n) \sim (y_1, \ldots, y_n). \\ \iff & \text{Two sets } \{x_1, \ldots, x_n\} \text{ and } \{y_1, \ldots, y_n\} \text{ are identical.} \end{aligned}$$

Let $\mathcal{B}^n$ denote a standard product σ-algebra on X^n, and let $\mathcal{B}^{(n)}$ be the σ-algebra on $X^{(n)}$ determined by the restriction (each element is different) and the equivalence relation $\sim$. This is one of the easiest way to construct a σ-algebra on the set spaces. These may be too fine to satisfy the uniqueness of σ-additive measures. Then, the structures of the set spaces may not be optimal for a constructively k-additive measure. At this step, we select one possible setting.

For a measurable set $A \in \mathcal{B}$, we define

$$A^{(n)} = \{(x_1, \ldots, x_n) \in X^{(n)} : x_j \in A,\ j \le n\}.$$

Moreover, for $U \in \mathcal{B}^{(j)}$ and $V \in \mathcal{B}^{(k)}$, we define

$$\begin{aligned} U(\times)V = \{(x_\ell)_{\ell=1}^{j+k} \in X^{(j+k)} &: (x_{\phi(1)}, \ldots, x_{\phi(j)}) \in U,\ (x_{\phi(j+1)}, \ldots, x_{\phi(j+k)}) \\ &\in V \quad \text{for some permutation } \phi \text{ of } (1, \ldots, j+k)\}. \end{aligned}$$

We remark that $U(\times)V$ is an element of $\mathcal{B}^{(j+k)}$ if $U \in \mathcal{B}^{(j)}$ and $V \in \mathcal{B}^{(k)}$.

We mainly deal with these set operations for infinite (measurable) sets. For finite sets, if their cardinality is very small, $A^{(n)}$ or $U(\times)V$ can be empty.

Now we are prepared to define constructive k-additivity.

Definition 1 (constructive k-additivity [5]). *A set function μ on $\mathcal{B}$ is constructively k-additive ($k \in \mathbb{N}$), or μ has constructive k-additivity, if there exists a σ-additive signed measure μ_j on $(X^{(j)}, \mathcal{B}^{(j)})$ for each $j = 1, 2, \ldots, k$ such that:*

$$\mu(A) = \sum_{j=1}^{k} \mu_j(A^{(j)})$$

for any $A \in \mathcal{B}$.

Consider the case where $\mathcal{B}$ is a finite family and X essentially is a finite set. Then, the classical Möbius transform and the inverse formula are available in this situation. Our generalization for the Möbius transform is a natural extension of Möbius transform for finite sub σ-algebras.

Definition 2 (generalized Möbius transform [6]). *Let μ be a set function on $\mathcal{B}$ and set $\mathcal{D}_n = \{\{A_1, A_2, \cdots, A_n\} : A_j \in \mathcal{B}, A_j \cap A_k = \emptyset, \quad j, k \le n, j \ne k\}$ for any $n \in \mathbb{N}$, that is, $\mathcal{D}_n$ is the family of all n disjoint measurable sets' combinations. A generalized Möbius transform $\{\nu_n\}$ of μ, which is a sequence of functions on $\mathcal{D}_n$, is defined as follows.*

(a) $\nu_1(A) = \mu(A), \quad \forall A \in \mathcal{B}$.

(b) $\nu_n(A_1, A_2, \ldots, A_n) = \mu(A_1 \cup A_2 \cup \cdots \cup A_n) - \left\{ \sum_{j=1}^{n-1} \sum_{1 \le i_1 < \cdots < i_j} \nu_j(A_{i_1}, \cdots, A_{i_j}) \right\},$

$\forall \{A_1, A_2, \ldots, A_n\} \in \mathcal{D}_n$.

We call $\nu_j()$ the j-order adjusting function for each $j \in \mathbb{N}$.

Using these concepts, another generalization for k-additivity is given as follows.

Definition 3 (formulaic k-additivity [6]). *Let μ be a set function on $\mathcal{B}$ and $\{\nu_n\}$ be the generalized Möbius transform of μ. μ satisfies formulaic k-additivity (or μ is a formulaic k-additive set function) if $\nu_j(\mathbf{A}) = 0$ for any $\mathbf{A} \in \mathcal{D}_j$ and $j \geq k+1$.*

Formulaic k-additivity satisfies the following equivalent conditions.

Proposition 1. *[6](Theorem 10).*
Let μ be a set function on $\mathcal{B}$, and let $\{\nu_n\}$ be its generalized Möbius transform. Then, the following are equivalent.

(a) μ is formulaic k-additive.
(b) $\nu_n = 0$ for any $n > k$.
(c) Assume that $(A_1, \cdots, A_{k-1}, B_1), (A_1, \cdots, A_{k-1}, B_2) \in \mathcal{D}_k$, and $B_1 \cap B_2 = \emptyset$. Then

$$\nu_k(A_1, \cdots, A_{k-1}, B_1 \cup B_2) = \nu_k(A_1, \cdots, A_{k-1}, B_1) + \nu_k(A_1, \cdots, A_{k-1}, B_2).$$

Remark. Using condition (b), a formulaic k-additive set function is also formulaic k'-additive for any $k' \geq k$.

3 Formulaic k-additivity of a constructively k-additive set function

In this section, we prove the formulaic k-additivity of a constructively k-additive set function.

Proposition 2. *Let μ_n be a σ- additive signed measure on $(X^{(n)}, \mathcal{B}^{(n)})$, and μ be a set function on $\mathcal{B}$ defined by*

$$\mu(A) = \mu_n(A^{(n)}).$$

Then, for each $k \leq n$, ν_k is represented by

$$\nu_k(A_1, \ldots, A_k) = \sum_{\substack{i_1 + \cdots + i_k = n \\ 1 \leq i_1, \ldots, i_k}} \mu_n(A_1^{(i_1)}(\times) \cdots (\times) A_k^{(i_k)}).$$

Proof. We will prove this property by induction on k. For $k = 1$, this property is easily given by the fact that $\nu_1(A_1) = \mu(A_1) = \mu_n(A^{(n)})$. Assume the assertion for $k \leq k_0 - 1$. Then, we have

$$
\begin{aligned}
&\nu_{k_0}(A_1, \cdots, A_{k_0}) \\
&= \mu(A_1 \cup \cdots \cup A_{k_0}) - \sum_{j=1}^{k_0-1} \sum_{1 \leq \ell_1 < \cdots < \ell_j \leq k_0} \nu_j(A_{\ell_1}, \ldots, A_{\ell_j}) \\
&= \mu_n((A_1 \cup \cdots \cup A_{k_0})^{(n)}) - \sum_{j=1}^{k_0-1} \sum_{1 \leq \ell_1 < \cdots < \ell_j \leq k_0} \nu_j(A_{\ell_1}, \ldots, A_{\ell_j}) \\
&= \sum_{j_1 + \cdots + j_{k_0} = n} \mu_n(A_1^{(j_1)}(\times) \cdots (\times) A_{k_0}^{(j_{k_0})}) \\
&\quad - \sum_{j=1}^{k_0-1} \sum_{1 \leq \ell_1 < \cdots < \ell_j \leq k_0,} \sum_{i_1 + \cdots + i_j = n} \mu_n(A_{\ell_1}^{(i_1)}(\times) \cdots (\times) A_{\ell_j}^{(i_j)}) \qquad (1) \\
&= \sum_{j_1 + \cdots + j_{k_0} = n} \mu_n(A^{(j_1)}(\times) \cdots (\times) A^{(j_{k_0})}) \\
&\quad - \sum_{i_1 + \cdots + i_{k_0} = n, \exists j, i_j = 0} \mu(A_{\ell_1}^{(i_1)}(\times) \cdots (\times) A_{\ell_{k_0}}^{(i_{k_0})}) \\
&= \sum_{\substack{j_1 + \cdots + j_{k_0} = n \\ 1 \leq j_1, \cdots, j_{k_0}}} \mu_n(A_1^{(j_1)}(\times) \cdots (\times) A_{k_0}^{(j_{k_0})})
\end{aligned}
$$

We obtain formula (1) by the induction hypothesis. This implies the assertion for $k = k_0$ and concludes the proof. □

Theorem 1. *For any $n \in \mathbb{N}$, a constructive n-additive set function satisfies formulaic n-additivity.*

Proof. We consider the case of

$$\mu(A) = \mu_n(A^{(n)}).$$

Using Proposition 2, for any disjoint sets $A_1, \ldots, A_{n-1}, B_1, B_2 \in \mathcal{B}$, we have:

$$
\begin{aligned}
&\nu_n(A_1, \ldots, A_{n-1}, B_1 \cup B_2) = \mu_n(A_1(\times) \cdots (\times) A_{n-1}(\times) B_1 \cup B_2) \\
&= \mu_n(A_1(\times) \cdots (\times) A_{n-1}(\times) B_1) + \mu_n(A_1(\times) \cdots (\times) A_{n-1}(\times) B_2) \\
&= \nu_n(A_1, \ldots, A_{n-1}, B_1) + \nu_n(A_1, \ldots, A_{n-1}, B_2).
\end{aligned}
$$

Then, the above formula follows the formulaic n-additivity of μ using Proposition 1.

Generally, μ can be represented by

$$\mu = \sum_{k=1}^{n} \mu_k(A^{(k)})$$

using signed measures μ_k on $(X^{(k)}, \mathcal{B}^{(k)})$. By the above arguments, $A \mapsto \mu_k(A^{(k)})$ satisfies formulaic k-additivity. Then these are formulaic n-additive since $k \leq n$ (Recall the remark after Proposition 1.) □

4 *k*-additivity of distorted measure

A set function μ on $\mathcal{B}$ is a distorted measure if there is a probability measure m on $(X, \mathcal{B})$ and non-decreasing continuous function f on $\mathbb{R}$ with $f(0) = 0$ such that

$$\mu(A) = f(m(A))$$

for any $A \in \mathcal{B}$. The non-decreasing function f is called "a distortion function". A distorted measure is monotone measure,that is , $\mu(A) \leq \mu(B)$ if $A \subset B$. If a distorted measure is formulaic k-additive measure, the distortion function must be a polynomial.

Proposition 3. *[6] (Theorem 17)*
Let m be a probability measure on $(X, \mathcal{B})$. Let f be the distortion function of a distorted measure $\mu(A) = f(m(A))$ $(A \in \mathcal{B})$. We assume that, for any $t, s \in \{m(A) : A \in \mathcal{B}\}$ and $A \in \mathcal{B}$ with $m(A) = t$, there exists $B \subset A$ such that $\mu(B) = s$ (this property is called "strong Darboux property"). Then, μ is formulaic k-additive if and only if f is a k-degree polynomial.

In the case where the distortion function of a distorted measure μ is a k-degree polynomial, then μ is constructively k-additive. This property was essentially proven by R. Mesiar [4], and we explain this using our notations.

Proposition 4. *(R. Mesiar [4])*
Let m be a positive finite σ-additive measure on $(X, \mathcal{B})$ and μ be a distorted measure given by $\mu(A) = f(m(A))$ $(A \in \mathcal{B})$ using a distortion function f. If f is a k-th degree polynomial, then μ is constructively k-additive.

Proof. We only need to prove this proposition for $f(x) = x^k$. The product measure m^k (defined on $(X^k, \mathcal{B}^k)$) can be easily reduced to the set space $(X^{(k)}, \mathcal{B}^{(k)})$, which concludes the proof. □

Summing up the propositions in this section, we arrive at the following theorem.

Theorem 2. *Let μ be a distorted measure on $(X, \mathcal{B})$ and $k \in \mathbb{N}$. Then, μ is constructively k-additive if and only if μ is formulaic k-additive.* □

5 *k*-additivity in a general case

We have proved that any constructively k-additive set functions are formulaic k-additive. In this section, we consider whether the reverse statement is true.

First, we consider the case where $\mathcal{B}$ is a finite family. As we mainly deal with infinite measurable spaces, the hypothesis "all one-point sets are measurable" must be removed, and the definition of the n-th power set $A^{(n)}$ should be modified. For an element x of X, let $[x]$ denote the smallest measurable set including x. Then, the definition of $A^{(n)}$ is modified by:

$$A^{(n)} = \{(x_1, x_2, \dots, x_n) \in A^n : \text{ if } j \neq j',\ x_{j'} \notin [x_j]\}.$$

Remark 1. Let $\mathcal{B}$ be a finite σ-algebra. Then, there exists a family of atoms $\mathbb{D} = \{D_1, D_2, \cdots, D_L\}$, that is, $\mathbb{D} \subset \mathcal{B}$ is a disjoint family satisfying $\mathcal{B} = \sigma(\mathbb{D})$. $\mathcal{B}^{(n)}$ $(n \in \mathbb{N})$ can be expressed as follows:

$$\mathcal{B}^{(n)} = \sigma\left(\{D_{i_1}(\times)\cdots(\times)D_{i_n} \ : \ 1 \leq i_1 < \cdots < i_n \leq n\}\right).$$

$\{D_{i_1}(\times)\cdots(\times)D_{i_n} \ : \ 1 \leq i_1 < \cdots < i_n \leq n\}$ is the family of all atoms in $\mathcal{B}^{(n)}$.

Proposition 5. *Let $(X, \mathcal{B})$ be a measurable space with the finite σ-algebra $\mathcal{B}$. Assume that a set function μ is formulaic k-additive. Then, for each $j \leq k$, we can construct a measure μ_j on each set space $(X^{(j)}, \mathcal{B}^{(j)})$ satisfying*

$$\mu(A) = \sum_{j=1}^{k} \mu_j(A^{(j)}).$$

Proof. Because the σ-algebra $\mathcal{B}$ is a finite set family, there exists a finite partition $\{D_j\}_{j=1}^n$ of X such that $\mathcal{B} = \sigma(\{D_j\}_{j=1}^n)$. Then, any $A \in \mathcal{B}$ can be represented by

$$A = \bigcup_{\ell=1}^{L} D_{i_\ell}, \quad 1 \leq i_1 < \cdots < i_L \leq n.$$

Let $j \leq k$ be a positive integer. Then, $\mathcal{B}^{(j)}$ (the σ-algebra of the set space $X^{(j)}$) can be represented as

$$\mathcal{B}^{(j)} = \sigma\left(\{D_{i_1}(\times)\cdots(\times)D_{i_j} : 1 \leq i_1 < \cdots < i_j \leq n\}\right),$$

and an element in $\mathcal{B}^{(j)}$ can be represented by a finite union of some subset of $\{D_{i_1}(\times)\cdots(\times)D_{i_j} : 1 \leq i_1 < \cdots < i_j \leq n\}$.

Without loss of generality, we assume that $\ell_i = i$ for each $i \leq L$. Then, the j-th power set is given by

$$A^{(j)} = \bigcup_{1 \leq \ell_1 < \ell_2 < \cdots < \ell_j \leq L} D_{\ell_1}(\times)\cdots(\times)D_{\ell_j}.$$

Let $\{\nu_j\}$ be a Möbius transform of the set function μ. Then, $\nu_j = 0$ for any $j \geq k+1$. We define a measure μ_j on $(X^{(j)}, \mathcal{B}^{(j)})$ by

$$\mu_j(D_{i_1}(\times)\cdots(\times)D_{i_j}) = \nu_j(D_{i_1}, \cdots, D_{i_j})$$

for each $(i_1, \ldots, i_j)$ $(1 \le i_1 < \cdots < i_j \le n)$.

$$\begin{aligned}\mu(A) &= \mu(\bigcup_{\ell=1}^{L} D_\ell) \\ &= \sum_{j=1}^{k} \sum_{1\le i_1<\cdots<i_j\le L} \nu(D_{i_1}, \cdots, D_{i_j}) \\ &= \sum_{j=1}^{k} \sum_{1\le i_1<\cdots<i_j\le L} \mu_j(D_{i_1}(\times)\cdots(\times)D_{i_j}) \\ &= \sum_{j=1}^{k} \mu_j(A^{(j)}).\end{aligned}$$

Thus we have proved the proposition. □

For further discussion, we will give some notations. Recall that $\mathcal{D}_j$ $(j \in \mathbb{N})$ denotes the family of j-disjoint measurable sets. Let $\overline{D}$ be an element of $\mathcal{D}_j$ $(\overline{D} = \{D_1, \cdots, D_j\} \in \mathcal{D}_j)$. Set

$$\nu_j(\overline{D}) = \nu(D_1, \cdots, D_j),$$

and

$$(\times\overline{D}) = D_1(\times)\cdots(\times)D_j \subset X^{(j)}.$$

Now, we give the following definitions.

Definition 4. *Let μ be a set function on $\mathcal{B}$ and $\{\nu_j\}$ be its Möbius transform. We define the j-th order total variation of ν_j as follows.*

$$\|\nu_j\| = \sup\{\sum_{\ell=1}^{L} |\nu(\overline{D_\ell})| : L \in \mathbb{N}, \overline{D_\ell} \in \mathcal{D}_j, \ell \le L, (\times\overline{D_\ell}) \cap (\times\overline{D_{\ell'}}) = \emptyset \text{ if } \ell = \ell'\}.$$

Then, μ is said to have k-th order bounded variation if $\|\nu_j\| < \infty$ for any $j \le k$.

Next, we define the fine continuity of μ at $\emptyset$.

Definition 5. *Let μ be a set function on $\mathcal{B}$ and $\{\nu_j\}$ be its Möbius transform. Then, the j-th adjusting function ν_j has fine continuity at $\emptyset$ if, for any sequence $\{\{\overline{D_i^{(\ell)}}\}_{i=1}^{N_\ell}\}_{\ell=1}^{\infty}$ of the disjoint finite set family in $\mathcal{D}_j$ satisfying*

$$\bigcup_{i=1}^{N_\ell} (\times\overline{D_i^{(\ell)}}) \searrow \emptyset \quad \text{as} \quad \ell \to \infty,$$

ν_j satisfies

$$\lim_{\ell\to\infty} \sum_{i=1}^{N_\ell} |\nu_j(\overline{D_i^{(\ell)}})| = 0.$$

Moreover, μ is said to have k-order fine continuity at $\emptyset$ iff ν_j has fine continuity at $\emptyset$ for $j \le k$.

Using these concepts, we will show the constructive k-additivity of a formulaic k-additive set function. To prove the existence of the corresponding σ-additive measure, we use the following extension theorem. This is well known for a non-negative measure (see [7] for example); however, using standard additional arguments, the statement is valid in the following form.

Theorem 3. *[Caratheodory's extension theorem] Let $\mathcal{A}$ be an algebra on X and μ be a finitely additive signed measure on $(X, \mathcal{A})$. Assume that*

$$\sup \left\{ \sum_{i=1}^{n} |\mu(A_i)| : n \in \mathbb{N}, \{A_i\}_{i=1}^{n} \textit{is a disjoint family in } \mathcal{A} \right\} < \infty,$$

and

$$\sum_{i=1}^{N_n} \mu(A_i^{(n)}) \to 0 \quad (n \to \infty),$$

for an arbitrary sequence of the disjoint family $\{\{A_i^{(n)}\}_{i=1}^{N_n}\}_{n=1}^{\infty}$, $N_n \in \mathbb{N}$ for each $n \in \mathbb{N}$, for which the union $\bigcup_{i=1}^{N_n} A_i^{(n)}$ decreases to an empty set. Then, there exists an extension $\widetilde{\mu}$ on $(X, \sigma(\mathcal{A}))$ satisfying $\widetilde{\mu}(A) = \mu(A)$ for any $A \in \mathcal{A}$.

Using these concepts and the above extension theorem, we arrive at the following theorem.

Theorem 4. *Let $\mathcal{A}$ be a countable algebra, and assume that $\mathcal{B} = \sigma(\mathcal{A})$. Let μ be a set function defined on $\mathcal{B}$ and k be a positive integer satisfying the following properties.*

(a) μ is a formulaic k-additive set function on $(X, \mathcal{B})$.
(b) μ has k-oder bounded variation.
(c) μ has k-oder fine continuity at $\emptyset$.
(d) μ is continuous from below and above.

Then, μ is constructively k-additive on $(X, \sigma(\mathcal{A}))$.

Proof. For a countable algebra $\mathcal{A}$, we can construct a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ of increasing finite algebras, which satisfies $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$. Then, $\mathcal{A}^{(j)} = \bigcup_{n=1}^{\infty} \mathcal{A}_n^{(j)}$ for any $j \leq k$. Using Proposition 5, for each $n \in \mathbb{N}$, there exist σ-additive measures $\mu_j^{(n)}$ on $(X^{(j)}, \mathcal{A}_n^{(j)})$ $(j \leq k)$ satisfying

$$\mu(A) = \sum_{j=1}^{k} \mu_j^{(n)}(A^{(j)}), \quad A \in \mathcal{A}_n.$$

Let us define an extension $\widetilde{\mu_j^{(n)}}$ of $\mu_j^{(n)}$ as follows.

$$\widetilde{\mu_j^{(n)}}(U) = \begin{cases} \mu_j^{(n)}(U) & \text{if } U \in \mathcal{A}_n^{(j)} \\ 0 & \text{if } U \notin \mathcal{A}_n^{(j)}, \end{cases} \quad U \in \mathcal{A}^{(j)}.$$

Then, for each $j \leq k$ and $U \in \mathcal{A}$, the sequence $\{\widetilde{\mu_j^{(n)}}(U)\}_{n=1}^{\infty}$ is bounded by the assumption (b). In general, a bounded sequence has a convergent sub-sequence. Thus, by countably selecting sub-sequences many times, there exists a sub-sequence $\mu_j^{(n_\ell)}$ such that $\{\mu_j^{(n_\ell)}(U)\}_\ell$ converges for any $U \in \mathcal{A}$ and $j \leq k$. Thus, we define a set function

$$\mu_j(U) = \lim_{\ell\to\infty} \mu_j^{(n_\ell)}(U).$$

Any element $U \in \mathcal{A}$ belongs to $\mathcal{A}_n$ for a sufficiently large $n \in \mathbb{N}$ and $\mu_j^{(n)}$ is finitely additive on $\mathcal{A}_n$. Then, the limit μ_j is also finitely additive on $\mathcal{A}$. Assumptions (b) and (c) imply that μ_j satisfies the assumptions of Theorem 3 and μ_j can be extended on $(X^{(j)}, \mathcal{B}^{(j)})$ as a σ-additive signed measure for $j \leq k$.

On a finite σ algebra constructive k-additivity is derived from formulaic k-additivity. Thus, constructive k-additivity is valid on $\mathcal{A}$, and using the continuity from above and below (condition (d)), this property can be extend to the minimal monotone class including $\mathcal{A}$. It is well known that this class is same with $\sigma(\mathcal{A})$ (see [8] for example). Then, we obtain constructive k-additivity on $(X, \sigma(\mathcal{A}))$. □

6 Conclusion

In this study, we discussed the relation between constructive and formulaic k-additivity. A constructively k-additive set function is always formulaic k-additive. A distorted measure is constructively k-additive if and only if it is formulaic k-additive. We defined "k-order bounded variation" and "fine continuity at $\emptyset$" for a set function, and using these concepts, we gave a sufficient condition for constructive k-additivity for a formulaic k-additive measure.

Constructive k-additivity must be useful for further arguments. The existence of a σ-finite measure is important, for example, to construct an L_p-theory for functional analysis on non additive monotone measure spaces. There remain several problems for the advance of these concepts. To show the uniqueness of the σ-additive measure on the set spaces, to make the structure of σ-algebra of the set spaces clear, and other detailed problems. We have to try to solve these problems.

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