Null Additivization of a Monotone Measure on a Finite Set

Ryoji Fukuda[1], Aoi Honda[2], and Yoshiaki Okazaki[3]

[1] Oita University, 700 Dan-noharu, Oita city, Oita 870-1192, JAPAN
rfukuda@oita-u.ac.jp
[2] Kyushu Institute of Technology, 680-4 Kawazu, Iizuka city, Fukuoka 820-8502, JAPAN
aoi@ai.kyutech.ac.jp
[3] Fuzzy Logic Systems Institute, 680-41 Kawazu, Iizuka city, Fukuoka 820-0067, JAPAN
okazaki@flsi.or.jp

Abstract. Massless points with respect to a monotone measure are inconspicuous, however, they sometimes play important roles. In this article, we try to treat such roles along with standard elements of the monotone measure space. By using Möbius transform, every inconspicuous role can be expressed by a certain element. We provide a method to construct a null additive space in which inconspicuous elements and standard elements in the given monotone measure space are mixed. In this space, a function on the original space corresponds to a certain function on the null additive space, and Choquet integral values are preserved under this translation.

Keywords: Fuzzy measure theory, Set functions, measures and integrals with values in ordered spaces.

1 Introduction

In this study, we discuss the null additivity of monotone measures, or set functions more generally. For a set function μ defined on a measurable space $(X, \mathcal{B})$, we always assume that $\mu(\emptyset) = 0$. We define $A \in \mathcal{B}$ as a μ-null set (or simply a null set) if any measurable subset B of A satisfies $\mu(B) = 0$. We define that μ is null additive if $\mu(B) = \mu(B \cup A)$ for any null set A and a measurable set $B \in \mathcal{B}$, and that μ is weakly null additive if $A \cup B$ is a μ-null set for any null sets $A, B \in \mathcal{B}$. In the case where μ is a monotone measure, the null additivity is defined in [1] or [2], and these definitions are delicate and differ slightly from study to study.

The concept of null and weak null additivities are important in analyses of measurable function on monotone measure spaces. For example, considering the relation $\mu(\{x : f(x) \neq g(x)\}) = 0$ (we denote it briefly by $\mu(\{f \neq g\}) = 0$ in the sequel) for a pair of measurable functions f, g, a monotone measure μ is weakly null additive if and only if this relation is an equivalent relation. Moreover, under the condition of null-continuity from below, the weak null additivity implies the existence of completion of the σ-algebra $\mathcal{B}$, and the completion is unique when μ is null additive. ([3],[4]). For a non-discrete monotone measure space, equivalent conditions for the null and weak null additivities can be described by the generalized Möbius transform (for the definition of Möbius transform, see [5]; for the equivalence conditions, [4]). Using the relation between classical and generalized Möbius transforms, which we show in this study, we describe conditions for the null and weak null additivities using the classical Möbius transform.

A set function with constructive k-additivity ($k \in \mathbb{N}$), the definition of which was given in [6], can be described using a signed measure on the space of finite subsets with cardinalities not more than k. In the case where X is a finite set, any set functions are (constructively) k-additive for some k ($k \leq |X|$). Each point mass of the above measure with respect to a point A (the measure is defined on a set family) is equivalent with the corresponding Möbius transform τ_A. Then, if there is a massless point $a \in X$, some influence factor may be represented using a Möbius transform with respect to some finite subset of X. Thus, for an arbitrary monotone measure space, we aim to construct a new non-additive measure space with no massless point by replacing some points with suitable influence factors, each of which has a one-to-one correspondence with some subset of X.

The new non-additive measure space inherit certain properties of the original monotone measure space. As the first step, for a given monotone measure space (X, μ) (X is a finite set and μ is a monotone measure) and a nonnegative function f on X, we aim to construct a non-additive measure space$(\widetilde{X}, \widetilde{\mu})$ and a function $\widetilde{f}$ on $\widetilde{X}$, for which two Choquet integrals $\int^{Ch} f d\mu$, $\int^{Ch} \widetilde{f} d\widetilde{\mu}$ are equivalent. Moreover, the transformed $(\widetilde{X}, \widetilde{\mu})$ and a function $\widetilde{f}$ also have same distribution functions. Then, for all distribution function type integrals, the Sugeno integrals or the Shilkret integrals, two integral values are equivalent.

2 Basic Properties

Let $(X, \mathcal{B})$ be a measurable space and μ be a set function ($\mu(\emptyset) = 0$). We define the null set, null additivity, and null additivity as follows.

Definition 1. (a) *$A \in \mathcal{B}$ is a μ-null set (simply null set) if $\mu(B) = 0$ for any $B \subset A, B \in \mathcal{B}$.*
(b) *μ is null additive if $\mu(B) = \mu(B \cup A)$ for any null set A and $B \in \mathcal{B}$.*
(c) *μ is weakly null additive if $\mu(B \cup A) = 0$ for any null sets A and B.*

The generalized Möbius transform $\tau(\cdots)$ is defined in Definition 2. Consider a partition $\mathbb{D} = \{D_j\}_{j=1}^n$ of a general measurable space, then a set function can be regarded as a set function defined on the n-point space $\mathbb{D}$. Under this restriction, the generalized Möbius transform coincides with the classical one.

Definition 2. *([5])*

(a) $\mathcal{D} = \{\mathbb{D} = \{D_j\}_{j=1}^n, n \in \mathbb{N}, D_j \in \mathcal{B}, D_j \cap D_k = \emptyset, \forall j, k \leq n,\ j \neq k\}$.

(b) $\tau(\mathbb{D}) = \tau(D_1, \cdots, D_n)$, $(\mathcal{D} \to \mathbb{R})$ *is defined by inductively:*

1. $\tau(\{D\}) = \mu(D)$ *for any* $D \in \mathcal{B}$.

2. $\tau(\mathbb{D}) = \mu(\bigcup_{D \in \mathbb{D}} D) - \sum_{\mathbb{D}' \subsetneq \mathbb{D}} \tau(\mathbb{D}')$.

The null and weak null additivities are described by the generalized Möbius transform.

Proposition 1. *([4]) Let* $(X, \mathcal{B}, \mu)$ *be a non-additive measure space.*
(a) μ *is null additive if and only if*

$$\mathbb{D} \in \mathcal{D}, \exists A \in \mathbb{D},\ A \text{ is a null set.} \Rightarrow \tau(\mathbb{D}) = 0.$$

(b) μ *is null additive if and only if*

$$\mathbb{D} \in \mathcal{D}, \forall A \in \mathbb{D},\ A \text{ is a null set.} \Rightarrow \tau(\mathbb{D}) = 0.$$

Originally, the above proposition was proven for the case in which μ is a monotone measure. The same method is valid for this case.

The classical Möbius transform for a set function on a finite set is well known (see for example [7]) and it was used in various situations. In the case where X is a finite set, we consider the classical Möbius transform $\{\tau_A\}_{A \subset X}$ of a set function $\mu : (2^X \to \mathbb{R})$. The generalized Möbius transform can be represented by the classical one.

Proposition 2. *Let* X *be a finite set,* μ *be a set function with* $\mu(\emptyset) = 0$, τ *be the (generalized) Möbius transform, and* $\{\nu_B\}_{B \subset X}$ *be the classical Möbius transform. For an element* $\mathbb{D} = \{A_1, \cdots, A_n\}$, *we define the family of finite subsets* $\Gamma(A_1, \cdots, A_n)$ *as follows.*

$$\Gamma(A_1, \cdots, A_n) = \{B \subset \bigcup_{j \le n} A_j : \forall j \le n, A_j \cap B \neq \emptyset\}.$$

Then, we have:

(a) $\tau_A = \tau(\{a_1\}, \cdots, \{a_n\}), \quad A = \{a_1, \ldots, a_n\}$.
(b) $\tau(A_1, \cdots, A_n) = \sum_{B \in \Gamma(A_1, \cdots, A_n)} \tau_B$

Proof. (a) We can describe the definition of classical Möbius transform by a similar method with Definition 1, and this may be easily verified using the above definition. It was also checked in [5]. (b) We prove this formula by induction on the cardinality $|\mathbb{D}| = n$. For a set $A \subset X$, by the definition of the Möbius transform,

$$\mu(A) = \sum_{B \subset A} \tau_B.$$

Then, this proves the case $n = 1$ because $\tau(\{A\}) = \mu(A)$, and $\Gamma(A) = \{B : B \subset A,\ B \neq \emptyset\}$.

Assume this formula when n is not more than $n_0 - 1$. For any $n \in \mathbb{N}$ and a disjoint set family $\{A_1, \cdots, A_n\}$, the following formula was proven in [5] for any n.

$$
\begin{aligned}
&\tau(A_1, \cdots, A_{n-2}, A_{n-1} \cup A_n) \\
&= \tau(\cdots, A_{n-2}, A_{n-1}) + \tau(\cdots, A_{n-2}, A_n) + \tau(\cdots, A_{n-2}, A_{n-1}, A_n).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\tau(\cdots, A_{n-2}, A_{n-1}, A_n) \\
&= \tau(\cdots, A_{n-2}, A_{n-1} \cup A_n) - \tau(\cdots, A_{n-2}, A_{n-1}) - \tau(\cdots, A_{n-2}, A_n).
\end{aligned}
$$

Set

$$\mathbb{D}_0 = \{\cdots, A_{n_0-2}, A_{n_0-1} \cup A_{n_0}\}, \quad \mathbb{D}_1 = \{\cdots, A_{n_0-2}, A_{n_0-1}\},$$

$$\mathbb{D}_2 = \{\cdots, A_{n_0-2}, A_{n_0}\}, \quad \mathbb{D}_3 = \{\cdots, A_{n_0-2}, A_{n_0-1}, A_{n_0}\}.$$

Then,

$$\tau(\mathbb{D}_3) = \sum_{B \in \Gamma(\mathbb{D}_0)} \tau_B - \sum_{B \in \Gamma(\mathbb{D}_1)} \tau_B - \sum_{B \in \Gamma(\mathbb{D}_2)} \tau_B.$$

For an element of $F \in \Gamma(A_1, \cdots, A_{n_0-2}, A_{n_0-1} \cup A_{n_0})$ satisfies one and only one of the following (1) $\sim$ (3).

(1) $F \cap A_{n_0-1} \neq \emptyset$ and $F \cap A_{n_0} \neq \emptyset$, that is, $F \in \Gamma(A_1, \cdots, A_{n_0-2}, A_{n_0-1}, A_{n_0})$.
(2) $F \cap A_{n_0-1} \neq \emptyset$ and $F \cap A_{n_0} = \emptyset$, that is, $F \in \Gamma(A_1, \cdots, A_{n_0-2}, A_{n_0-1})$.
(3) $F \cap A_{n_0-1} = \emptyset$ and $F \cap A_{n_0} \neq \emptyset$, that is, $F \in \Gamma(A_1, \cdots, A_{n_0-2}, A_{n_0})$.

Moreover, $\Gamma(\mathbb{D}_1)$, $\Gamma(\mathbb{D}_2)$, and $\Gamma(\mathbb{D}_3)$ are disjoint from each other. This implies that

$$\sum_{B \in \Gamma(\mathbb{D}_0)} \tau_B - \sum_{B \in \Gamma(\mathbb{D}_1)} \tau_B - \sum_{B \in \Gamma(\mathbb{D}_2)} \tau_B = \sum_{B \in \Gamma(\mathbb{D}_3)} \tau_B,$$

and this concludes the proof. □

The above proposition asserts that generalized Möbius transform can be expressed by classical one when $|X| < \infty$. The conditions for the null and weak null additivities are expressed as follows.

Proposition 3. *Let X be a finite set and μ be a set function defined on 2^X. Then, we have the following.*
(a) μ is null additive if and only if

$$A \subset X,\ \exists x \in A,\ \mu(\{x\}) = 0, \Rightarrow \tau_A = 0. \tag{1}$$

(b) μ is null additive if and only if

$$A \subset X,\ \forall x \in A,\ \mu(\{x\}) = 0, \Rightarrow \tau_A = 0.$$

Proof. (a) Let $\mathbb{D} = \{A_1, \cdots, A_n\}$ be an arbitrary element of $\mathcal{D}$. Then, by Proposition 1, we have only to prove that

$$\mu(A_1) = 0 \ \Rightarrow \ \tau(\{A_1, \cdots, A_n\}) = 0, \tag{2}$$

using the condition (1). Because the condition (2) clearly implies (1), then we have that conditions (1) and (2) are equivalent.

By Proposition 2,

$$\tau(\mathbb{D}) = \sum_{B \in \Gamma(\mathbb{D})} \tau_B.$$

For any element $B \in \Gamma(\mathbb{D})$, there exist $a \in B \cap A_1$. By the assumption of (a), we have $\tau_B = 0$ for each $B \in \Gamma(\mathbb{D})$. Hence, $\tau(\mathbb{D}) = 0$.

(b) We need only to prove that $\tau(\{A_1, \cdots, A_n\}) = 0$ ($\mathbb{D} = \{A_1, \cdots, A_n\} \in \mathcal{D}$) if A_k is a null set for each $k \leq n$, under the assumption of (b).

To do so, we use the Proposition 2 again:

$$\tau(\mathbb{D}) = \sum_{B \in \Gamma(\mathbb{D})} \tau_B.$$

By the definition of $\Gamma(\mathbb{D})$, any element $B \in \Gamma(\mathbb{D})$ satisfies

$$B \subset \bigcup_{k \leq n} A_k.$$

Hence, each point $b \in B$ satisfies $\mu(\{b\}) = 0$ because $b \in A_k$ for some $k \leq n$ and A_k is a null set for any $k \leq n$. Then, we also have $\tau_B = 0$ for any $B \in \Gamma(\mathbb{D})$, and this concludes the proof. □

3 Influence Factor

By the arguments in the previous section, a set function on a finite set is null additive if there are no massless points. The purpose of this section is to construct a null additive non-additive measure space, by removing massless points and adding some influence elements.

Definition 3. *Let X be a finite set and μ be a set function on 2^X. Then, $A \subset X$ is an influence factor if there exists $\exists a \in A$ such that*

(a) $\mu(\{a\}) = 0$,
(b) $\tau_A \neq 0$, *and*
(c) $a \in B \subsetneq A \ \Rightarrow \ \tau_B = 0$.

N denotes the family of all massless points, and $\mathcal{A}$ denotes the family of all influence factors. We consider an influence element ι_A corresponding to an influence factor $A \in \mathcal{A}$.

For each $B \subset X$, set

$$I_B = \{A | A \in \mathcal{A}, \ A \subset B\},$$
$$\mathbb{I} = \{B : B \cap N \neq \emptyset, \ B = (B \setminus N) \cup \bigcup_{A \in I_B} A\}.$$

Remarks: For a massless point a satisfies that $\tau_A = 0$ if $a \in A$, then we can remove a from X. Then, we assume that, for any point a, there exists A which satisfies $\tau_A \neq 0$ and $a \in A$. An influence factor is a minimal subset satisfying $a \in A$ and $\tau_A \neq 0$. However, in general, the relation between a massless point and an influence factor is not a one-to-one correspondence. The relation between "factor" and "element" is similar to that between "fuzzy set" and "membership function".

Proposition 4. *Let X be a finite set , μ be a monotone measure on 2^X, and A be an influence factor. Then, we have $\tau_A > 0$.*

Proof. Let $a \in A$ be a point satisfying the condition in Definition 3

$$\begin{aligned}\mu(A) &= \sum_{B \subsetneq A} \tau_B + \tau_A \\ &= \sum_{B \subsetneq A, a \notin B} \tau_B + \sum_{B \subsetneq A, a \in B} \tau_B + \tau_A \\ &= \mu(A \setminus \{a\}) + \tau_A.\end{aligned}$$

The last equality holds by the condition (c) in Definition 3. Using the monotonicity, we have $\tau_A > 0$. □

Proposition 5. *Let X be a finite set , μ be a monotone measure on 2^X. Then, we have the following.*

(a) If $\tau_B \neq 0$, then $B = (B \setminus N) \cup \bigcup_{A \in \mathcal{A}, A \subset B} A$.
(b) Let $B, B' \subset X$ be any subsets of X. Then $B = B'$ if and only if $(B \setminus N) \cup I_B = (B' \setminus N) \cup I_{B'}$.

Notation and Remark: Set $I_B = \{\iota_A, A \in \mathcal{A}, A \subset B\}$ and $\mathbb{I}$ is defined by $\mathbb{I} = \{B : B = (B \setminus N) \cup \bigcup_{A \in \mathcal{A}, A \subset B}\}$.

(a) of this proposition implies that $\tau_B = 0$ if $B \cap N \neq \emptyset$ and $B \notin \mathbb{I}$.

Proof. (a) Consider an element a satisfying $a \in B \cap N$. We prove that there exist $A \in \mathcal{A}$ such that $a \in A \subset B$. If all proper subsets $C \subsetneq A$ with $a \in C$ satisfies $\tau_C = 0$ then $B \in \mathcal{A}$. If there exist $C \subsetneq B$ with $a \in C$ and $\tau_C \neq 0$, by replacing B with C and iterating this process until we find an element of $\mathcal{A}$.

This property implies that, for all $a \in B \subset N$, there exist $C \in \mathcal{A}$ such that $a \in C \subset B$. Therefore, we have

$$B = (B \setminus N) \cup \left(\bigcup_{C \subset B, C \in \mathcal{A}} C \right),$$

and this concludes the proof of (a).

(b) As the "only if" part is clear, and we prove the contrapositive of the "if" part. Assume that $B \neq B$. If $B \cap N \neq B' \cap N$, clearly we have $(B \setminus N) \cup I_B \neq (B' \setminus N) \cup I_{B'}$. Then, we consider the case $B \cap N = B' \cap N$, and this implies $B \cap N \neq B' \cap N$. Assume that $\exists a \in (B \cap N) \setminus B'$. Then, there exists $A \in \mathcal{A}$ satisfying $a \in A \subset B$, because $B \supset \bigcup_{A \in \mathcal{A}, A \subset B} A = \bigcup_{A \in I_B} A$. Assuming that $A \in I_{B'}$ then $a \in A \subset B'$ contradicts to the assumption $a \notin B'$. Therefore, we have $I_B \neq I_{B'}$, and this concludes the proof. □

4 Construction of null additivization space.

Using the argument in previous sections, we define the null addivization of monotone measure space as follows.

Definition 4. *Let X be a finite set and μ be a monotone measure on 2^X. Set $\widetilde{X} = (X \setminus N) \cup \{\iota_A : A \in \mathcal{A}\}$. We define the transformed set function $\widetilde{\mu}$ by defining its Möbius transform $\widetilde{\tau}$ as follows.*

$$\begin{aligned} A \in \mathcal{A} \quad &\Rightarrow \quad \widetilde{\tau}_{\{A\}} = \tau_A, \\ B \subset X \setminus N \quad &\Rightarrow \quad \widetilde{\tau}_B = \tau_B, \\ B \in \mathbb{I}, B = \bigcup_{C \in I_B} C \quad &\Rightarrow \quad \widetilde{\tau}_{B \cup I_B} = 0, \\ B \in \mathbb{I}, B \neq \bigcup_{C \in I_B} C \quad &\Rightarrow \quad \widetilde{\tau}_{B \cup I_B} = \tau_B, \\ otherwise, U \subset \widetilde{X}\ \widetilde{\tau}_U = 0. \end{aligned}$$

Let $\widetilde{\mu}$ be the set function with the Möbius transform is $\widetilde{\tau}$. Then, $(\widetilde{X}, \widetilde{\mu})$ is the null additivization of (X, μ).

Definition 5. *Let (Y, ν) be a (not necessary monotone) non-additive measure space on a finite set, $\{\tau_B\}_{B \in 2^Y}$ be its Möbius transform, and f be a nonnegative function on Y. Then, we define a Choquet integral of f on (Y, ν) as follows.*

$$\int^{Ch} f d\nu = \sum_{B \in 2^Y} \min_{y \in B} f(y) \tau_B.$$

It is well-known that the above Choquet integral is identical with the standard version, when μ is a monotone measure. Then, the following theorem holds.

Theorem 1. *Let $(\widetilde{X}, \widetilde{\mu})$ and $\widetilde{f}$ be the null additivization of a monotone measure space (X, μ) and a nonnegative function f on X, which are given in Definition 4. Then, we have*

$$\int_X^{Ch} f d\mu = \int_{\widetilde{X}}^{Ch} \widetilde{f} d\widetilde{\mu}.$$

Proof.

By Proposition 5,

$$\begin{aligned} \int_X^{Ch} f d\mu &= \sum_{B \subset X} \min_{x \in B} f(x) \tau_B \\ &= \sum_{B \subset X \setminus N} \min_{x \in B} f(x) \tau_B + \sum_{B \in \mathbb{I}} \min_{x \in B} f(x) \tau_B \\ &= \sum_{B \subset X \setminus N} \min_{x \in B} f(x) \tau_B + \sum_{A \in \mathcal{A}} \min_{x \in A} f(x) \widetilde{\tau}_{\{A\}} + \sum_{B \in \mathbb{I}, \notin \mathcal{A}} \min_{y \in (B \setminus N) \cup I_B} f(x) \widetilde{\tau}_{\{(B \setminus N) \cup I_B\}} \\ &= \int_{\widetilde{X}}^{Ch} f d\widetilde{\mu}. \end{aligned}$$

□

Example 1. Set $X = \{a, b, c\}$ and define a monotone measure μ on X as follows.

$$\begin{aligned}
&\mu(\{a\}) = \mu(\{b\}) = 0,\ \mu(\{c\}) = 1,\\
&\mu(\{a,b\}) = 1, \mu(\{a,c\}) = 2,\ \mu(\{b,c\}) = 3,\\
&\mu(\{a,b,c\}) = 3.
\end{aligned}$$

Then, the Möbius transform is given by

$$\begin{aligned}
&\tau_{\{a\}} = \tau_{\{b\}} = 0,\ \tau_{\{c\}} = 1,\\
&\tau_{\{a,b\}} = \tau_{\{a,c\}} = 1,\ \tau_{\{b,c\}} = 2,\\
&\tau_{\{a,b,c\}} = -2.
\end{aligned}$$

In this case, the null addtivization is given by:

$$\begin{aligned}
\widetilde{X} &= \{c, \iota_{\{a,b\}}, \iota_{\{a,c\}}, \iota_{\{b,c\}}\}\\
&= \{c, \alpha, \beta, \gamma\},\\
N &= \{a, b\},\ \mathcal{A} = \{\alpha, \beta, \gamma\}.
\end{aligned}$$

Then, the translated Möbius transform is calculated as follows.

$$\begin{aligned}
&\widetilde{\tau}_{\{c\}} = 1, \widetilde{\tau}_{\{\alpha\}} = 1,\ \widetilde{\tau}_{\{\beta\}} = 1,\ \widetilde{\tau}_{\{\gamma\}} = 2,\\
&\widetilde{\tau}_{\{c,\alpha\}} = \widetilde{\tau}_{\{c,\beta\}} = \widetilde{\tau}_{\{c,\gamma\}} = 0,\\
&\widetilde{\tau}_{\{c,\alpha,\beta\}} = \widetilde{\tau}_{\{c,\beta,\gamma\}} = \widetilde{\tau}_{\{c,\alpha,\gamma\}} = 0,\\
&\widetilde{\tau}_{\{\alpha,\beta\}} = \widetilde{\tau}_{\{\beta,\gamma\}} = \widetilde{\tau}_{\{\alpha,\gamma\}} = 0,\\
&\widetilde{\tau}_{\{\alpha,\beta,\gamma\}} = 0,\\
&\widetilde{\tau}_{\{c,\alpha,\beta,\gamma\}} = (\tau_{a,b,c} =) - 2.
\end{aligned}$$

Let $\widetilde{\mu}$ be a set function defined by the above Möbius transform. We have $\widetilde{\mu}(\{\alpha, \beta, \gamma\}) = 4$ and $\widetilde{\mu}(\{c, \alpha, \beta, \gamma\}) = 3$. Hence, this set function is not monotone.

Consider a function f on X, $f(a) = 1, f(b) = 2, f(c) = 3$. Then,

$$\begin{aligned}
&f(\alpha) = f(a) \wedge f(b) = 1,\ f(\beta) = f(a) \wedge f(c) = 1,\\
&f(\gamma) = f(b) \wedge f(c) = 2.
\end{aligned}$$

$$\begin{aligned}
&\int^{Ch} f d\mu = \sum_{B \subset X} \min_{x \in B} f(x)\tau_B\\
&= f(c)\tau_{\{c\}} + (f(a) \wedge f(b))\tau_{\{a,b\}} + (f(a) \wedge f(c))\tau_{\{a,c\}} +\\
&\quad (f(b) \wedge f(c))\tau_{\{b,c\}} + (f(a) \wedge f(b) \wedge f(c))\tau_{\{a,b,c\}}\\
&= 3 \times 1 + 1 \times 1 + 1 \times 1 + 2 \times 2 + 1 \times (-2) = 7.
\end{aligned}$$

On the other hand, Choquet integral of translated function is calculated as follows.

$$\int^{Ch} f d\widetilde{\mu} = \sum_{U \subset \widetilde{X}} (\min_{x \in U} f(x)) \widetilde{\tau}_U$$
$$= f(c)\widetilde{\tau}_{\{c\}} + f(\alpha)\widetilde{\tau}_{\{\alpha\}} + f(\beta)\widetilde{\tau}_{\{\beta\}} + f(\gamma)\widetilde{\tau}_{\{\gamma\}} +$$
$$(f(\alpha) \wedge f(\beta) \wedge f(\gamma) \wedge f(c))\widetilde{\tau}_{\{c,\alpha,\beta,\gamma\}}$$
$$= 3 \times 1 + 1 \times 1 + 1 \times 1 + 2 \times 2 + 1 \times (-2) = 7.$$

Thus, two integral values are the same.

The translated set function in the above example is not monotone. Therefore, this example illustrates that a translated set function is not necessary monotone even if the original set function is monotone.

Lemma 1. *Let X be a finite set, μ be a monotone measure on 2^X, and f be a nonnegative function on X. $\widetilde{X}$, $\widetilde{\mu}$, and $\widetilde{f}$ are null addtivization of X, μ, and f respectively. Then,*

(a) *If $\int^{Ch} f d\mu = 0$, we have $f(x) = 0$ for any $x \in X \backslash N$, and $\min_{x \in A} f(x) = 0$ for any $A \in \mathcal{A}$, where N is the set of all massless points and $\mathcal{A}$ is the set of all influence factors.*

(b) $\mu(\{f(x) > r\}) = \widetilde{\mu}(\{\widetilde{f}(x) > r\})$ *for any $r > 0$.*

Remark: By Lemma 1 (b), the distribution function of f is same with that of $\widetilde{f}$. This implies that the Sugeno and Shilkret integrals take same values on the both spaces, where these integrals are defined on (not necessarily monotone) non-additive measure space using the distribution functions.

Proof. (a) Using the monotonicity of μ and the Choquet integral with respect to μ,

$$0 = \int^{Ch} f(x) d\mu \geq \int^{Ch} f(x) 1_{\{b\}} d\mu = f(b)\mu(\{b\}) \geq 0,$$

$$0 = \int^{Ch} f(x) d\mu \geq \int^{Ch} f(x) 1_A d\mu \geq (\min_{x \in A} f(x))\mu(A) \geq 0,$$

for any $b \in X \setminus N$, and $A \in \mathcal{A}$. We have $f(b) = 0$.

By the definition of $\mathcal{A}$, there exists $a \in A \cap N$ such that $\tau_B = 0$ if $a \in B \subsetneq A$. Then,

$$\mu(A) = \sum_{B \subset A \setminus \{a\}} \tau_B + \tau_A = \mu(A \setminus \{a\}) + \tau_A.$$

By Proposition 4 and $\mu(A \setminus \{a\}) \geq 0$, we have $\mu(A) > 0$, and this concludes the proof of (a).

(b) Fixing $r > 0$ and setting $A_r = \{x : f(x) > r\}$, we have:

$$\begin{aligned}\mu(A_r) &= \sum_{B \subset A_r} \tau_B \\ &= \sum_{B \subset A_r \setminus N} \tau_B + \sum_{B \subset A_r, B \in \mathbb{I}} \tau_B \\ &= \sum_{B \subset A_r \setminus N} \widetilde{\tau}_B + \sum_{A \subset A_r, A \in \mathcal{A}} \widetilde{\tau}_{\{A\}} + \sum_{B \subset A_r, B \notin \mathcal{A}, B \in \mathbb{I}} \widetilde{\tau}_{(B \setminus N) \cup I_B} \end{aligned} \tag{3}$$

We remark that:

- on $B \subset N$, $\widetilde{f}$ coincides with f,
- on a one point set $\{\iota_A\}, A \in \mathcal{A}$, $\widetilde{f}(\iota_A) = \min_{x \in A} f(x)$, then, $\iota_A \subset A_r$ if and only if $\{\iota_A\} \subset \{\widetilde{f}(y) > r\}$, and
- on $B \setminus N \cup I_B$, $y \in \{\widetilde{f}(y) > r\}$ if and only if $y \in B \setminus N$ or $y \in I_B$, $\widetilde{f}(y) = \min_{x \in y} f(x) > r$.

Hence, we have:

$$\begin{aligned}(3) &= \sum_{B \subset \{\widetilde{f}(y) > r\} \setminus \mathcal{A}} \widetilde{\tau}_B + \sum_{A \in \{\widetilde{f}(y) > r\} \cap \mathcal{A}} \widetilde{\tau}_{\{\iota_A\}} + \sum_{(B \setminus N) \cup I_B \subset \{\widetilde{f}(y) > r\}} \widetilde{\tau}_{(B \setminus N) \cup I_B} \\ &= \mu(\{\widetilde{f}(y) > r\}).\end{aligned}$$

□

5 Conclusion

In this study, we have analyzed relations between classical and non-discrete Möbius transforms, and given equivalence conditions for null and weak null additivity for set functions on a finite set. We have also defined a method to construct a null additive set function from a general monotone measure. Along with a translated function, in the new non-additive measure space, the distribution function and the distribution function type integrals remain unchanged under this translation.

Our construction of null additive spaces is based on the argument to express the target monotone measure by using a certain σ-additive signed measure on the family of finite subsets of target space. A similar situation can be found in [2], this provides an expression method to describe a monotone measure by σ-additive (non-negative) measure on some set family space. Our problem may be improved or developed by using this idea.

References

1. E. Pap, Null-Additive Set Functions, Mathematics and Its Applications, Kluwer Academic Publishers, 1995.
2. T. Murofushi, M. Sugeno, A theory of fuzzy measures: Representations, the choquet integral, and null sets, Journal of Mathematical Analysis and Applications 159 (1991) 532–549.
3. J. Kawabe, The topology on the space of measurable functions that is compatible with convergence in nonadditive measure, Fuzzy sets and systems(Available online) (2021).
4. R. Fukuda, A. Honda, Y. Okazaki, On relaxations for the subadditivity of a monotone measure, Preprint (2021).
5. A. Honda, R. Fukuda, Y. Okazaki, Non-discrete k-order additivity of a set function and distorted measure, Fuzzy sets and systems(Available online) (2021).
6. R. Fukuda, A. Honda, Y. Okazaki, Constructive k-additive measure and decreasing convergence theorems, Lecture Notes in Artificial Intelligence 12256 (2020) 104–116.
7. G. Rota, On the foundations of combinatorial theory, I. Theory of Möbius functions. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 2 (1964) 340–368.