Constructive set function and extraction of a $\boldsymbol{k}$-dimensional element

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Abstract. We define a constructive non-additive set function as a generalization of a constructively k-additive set function ($k \in \mathbb{N}$). First, we prove that a distortion measure is a constructive set function if the distortion function is analytic.
A signed measure on the extraction space represents a constructive set function. This space is the family of all finite subsets of the original space. In the case where the support of the measure is included in the subfamily whose element's cardinality is not more than k, the corresponding set function is constructively k-additive ($k \in \mathbb{N}$). For a general constructive set function μ, we define the k-dimensional element of μ, which is a set function, by restricting the corresponding measure on the extraction space to the above subfamily. We extract this k-dimensional element by using the generalized Möbius transform under the condition that σ-algebra is countably generated,

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1 Introduction

k-additivity of a set function on a finite set was introduced by M. Grabisch ([1],[2]). This reduces some complexities and was used in several situations (see, for example, [3], [4]). The Möbius transform of a set function μ is a set function, which gives a one-to-one correspondence between the original set function μ and the transformed set function $\{\tau_B\}_{B\subset X}$. (In this study, we assume that each set function μ satisfies $\mu(\emptyset) = 0$.) A set function μ is k-additive ($k \in \mathbb{N}$) if its Möbius transform $\{\tau_A\}_{A\subset X}$ satisfies $\tau_A = 0$ when the cardinality of A is greater than k.

Next, we consider a measurable space $(X, \mathcal{B})$, where X is not discrete in general, and $\mathcal{B}$ is a σ-algebra over X. The k-additivity of a set function on $(X, \mathcal{B})$ was proposed by R. Mesiar [5]. A signed measure on the product space of the original space represented a generalized k-additive set function. A generalized Möbius transform was defined, and formulaic k-additivity was defined in [7]. We call the former k-additivity constructive k-additivity.

We consider the family of all finite subsets of X whose element's cardinality is not more than k, which we call the extraction space with order k. This space is denoted by $X^{[\leq k]}$, and for each $A \in \mathcal{B}$, $A^{[\leq k]}$ denotes the family of all elements of $X^{[\leq k]}$ which are included in A. The constructive k-additivity was reformulated using $X^{[\leq k]}$, then μ is constructively k-additive if there is a signed (σ-additive) measure $\mu^{[\leq k]}$ on $X^{[\leq k]}$ which satisfies $\mu(A) = \mu^{[\leq k]}(A^{[\leq k]})$ for any $A \in \mathcal{B}$. We call the measure $\mu^{[\leq k]}$ the constructing measure. The measure $\mu^{[\leq k]}$ is uniquely determined in this formulation ([8]). On the other hand, for a set function μ on $(X, \mathcal{B})$, the Möbius transform was generalized as a function on the correction of all finite disjoint $\mathcal{B}$-measurable set families $\mathcal{D}$ ([8]). μ is formulaic k-additive iff $\tau(\mathbb{D}) = 0$ for any $\mathbb{D} \in \mathcal{D}$ satisfying $|\mathbb{D}| > k$, where $|\mathbb{D}|$ denotes the number of finite sets in $\mathbb{D} \in \mathcal{D}$. A constructively k-additive set function is always formulaic k-additive, and under certain conditions, a formulaic k-additive set function is constructively k-additive ([8]). The existence of the constructing measure is an advantage of a constructively k-additive set function, in various arguments. For example, the monotone decreasing convergence theorem for a Pan integral was proved when the corresponding fuzzy measure is constructively k-additive ([6]).

In this study, we define a constructive set function as a generalized constructively k-additive set function($k \to \infty$). A distorted measure is a set function μ described by $\mu(A) = f(\nu(A))$ using some finite measure ν and a function on $\mathbb{R}^+$ vanishing at 0. In section 3, we will prove that a distorted measure is constructive if the distortion function is analytic and satisfies certain additional conditions. Consider the case where f is a polynomial with degree k, then μ is constructively k-additive. In general, a constructively k-additive set function is formulaic k-additive, and in the case where a distorted measure is formulaic k-additive, the distortion function must be a polynomial with degree k if the corresponding finite measure satisfies "strong Darboux property" ([7]).

The constructing measure $\mu^{[*]}$ concerning a constructive set function μ is defined on $X^{[*]}$, which space is a set of all finite subsets of the original space. Then, logically, we can define the constructively k-additive set function by restricting the constructing measure to $X^{[k]}$, which is the subfamily of $X^{[*]}$ whose element's cardinality is k. We call this restriction the k-dimensional element of a set function. We propose a numerical extraction method for the k-dimensional element using the generalized Möbius transform of a constructive set function.

2 Constructive set function

Let $(X, \mathcal{B})$ be a measurable space, X is a set and $\mathcal{B}$ is a σ-algebra over X. We assume that each set function μ defined on $\mathcal{B}$ satisfies $\mu(\emptyset) = 0$. First, we define the space of finite subsets of X.

Definition 1. *Consider the space of all finite subsets of X and denote the space as follows.*

$$X^{[*]} = \left\{ \{x_j\}_{j=1}^n : n \in \mathbb{N},\ x_j \in X,\ j \le n \right\},$$

and for each $A \in \mathcal{B}$ we define

$$A^{[*]} = \left\{ \{x_j\}_{j=1}^n : n \in \mathbb{N},\ x_j \in A,\ j \le n \right\} \subset X^{[*]}.$$

We call the space $X^{[*]}$ an extraction space.

Let $k \in \mathbb{N}$ be an integer. Then we define the extraction space with order k $X^{[\le k]}$ and its subsets $A^{[\le k]}$ ($A \in \mathcal{B}$) as follows.

$$X^{[\le k]} = \left\{ \{x_j\}_{j=1}^n : n \le k,\ x_j \in X,\ j \le n \right\},$$

$$A^{[\le k]} = \left\{ \{x_j\}_{j=1}^n : n \le k,\ x_j \in A,\ j \le n \right\} \subset X^{[\le k]}.$$

We also define a space $X^{[k]}$. We use this concept to define a k-dimensional element of a set function.

$$X^{[k]} = \left\{ \{x_j\}_{j=1}^k :\ x_j \in X,\ j \le k \right\}, \quad (n = k).$$

$$A^{[k]} = \left\{ \{x_j\}_{j=1}^k :\ x_j \in A,\ j \le k \right\} \subset X^{[k]}.$$

Remark that $\{x_j\}_{j=1}^k$ is a set and $x_j \ne x_\ell$ if $j \ne \ell$.

Using the above $X^{[\le k]}$, constructive k additivity of a set function μ was defined [6] and we define constructive set function by generalizing $k \to \infty$.

Definition 2. *Let $(X, \mathcal{B})$ be a measurable space, we define σ-algebras $\mathcal{B}^{[*]}$ and $\mathcal{B}^{[\le k]}$ ($k \in \mathbb{N}$) as follows.*

$$\mathcal{B}^{[*]} = \sigma \left\{ A^{[*]} : A \in \mathcal{B} \right\},$$

$$\mathcal{B}^{[\le k]} = \sigma \left\{ A^{[\le k]} : A \in \mathcal{B} \right\}.$$

Then a set function μ on $\mathcal{B}$ is constructively k-additive if there exists a signed measure $\mu^{[\le k]}$ on $(X^{[\le k]}, \mathcal{B}^{[\le k]})$ such that

$$\mu(A) = \mu^{[\le k]}(A^{[\le k]}). \quad \forall A \in \mathcal{B},$$

and similary, μ is constructive if there exists a signed measure $\mu^{[*]}$ on $(X^{[*]}, \mathcal{B}^{[*]})$ such that

$$\mu(A) = \mu^{[*]}(A^{[*]}). \quad \forall A \in \mathcal{B}.$$

We call the measure $\mu^{[\le k]}$ or $\mu^{[*]}$ the constructing measure.

Next, we define a generalized Möbius transform.

Definition 3. *Let $(X, \mathcal{B})$ be a measurable space. We define a family of extraction bases $\mathcal{D}$ on $(X, \mathcal{B})$ as follows.*

$$\mathcal{D} = \{\mathbb{D} = \{D_j\}_{j=1}^n : n \in \mathbb{N}, D_j \in \mathcal{B}, j \leq n, D_j \cap D_k = \emptyset\ (j \neq k).\}$$

The size of each extraction basis $\mathbb{D} \in \mathcal{D}$ is defined by

$$|\mathbb{D}| = |\{D_1, \dots, D_n\}| = n,$$

and set

$$\cup\mathbb{D} = \bigcup_{j=1}^n D_j.$$

For a set function μ, we define a generalized Möbius transform τ of μ by induction. τ is a function on $\mathcal{D}$.

(a) $\tau(\mathbb{D}) = \tau(\{D_1\}) = \mu(D_1)$ $(|\mathbb{D}| = 1)$.
(b) *Assume that $\tau(\mathbb{D}')$ are defined for all $\mathbb{D}'$ satisfying $|\mathbb{D}'| < n$ $(n \in \mathbb{N})$. Then, for $\mathbb{D} \in \mathcal{D}$ with $|\mathbb{D}| = n$*

$$\tau(\mathbb{D}) = \mu(\cup\mathbb{D}) - \sum_{\mathbb{D}' \subsetneq \mathbb{D}} \tau(\mathbb{D}').$$

Note that the inclusion $\mathbb{D}' \subsetneq \mathbb{D}$ implies that $\mathbb{D}'$ is a strict subfamily of $\mathbb{D}$ as a family of disjoint measurable sets $(|\mathbb{D}'| < |\mathbb{D}|)$.

Fix an element $\mathbb{D}(= \{D_j^{(n)}\}_{j=1}^N) \in \mathcal{D}$, and consider a subfamily $\mathcal{D}_{\mathbb{D}} = \{\mathbb{D}' \subset \mathbb{D}, \mathbb{D}' \in \mathcal{D}\}$. We consider a set function μ, and set $\mu_{\mathbb{D}}$ as follows.

$$\mu_{\mathbb{D}}(\mathbb{D}') = \mu(\cup\mathbb{D}').$$

Let τ be the generalized Möbius transform of μ, $\tau_{\mathbb{D}}$ be its restriction to $\mathcal{D}_{\mathbb{D}}$. Then $\mu_{\mathbb{D}}$ is a set function defined on a finite set, and $\tau_{\mathbb{D}}$ is its classical Möbius transform. Thus, τ provides the classical Möbius transform for each finitely divided subfamily $\mathcal{D}'$. In this point of view, we call an element of $\mathcal{D}$ an extraction basis.

Next, we consider some subsets of the extraction space. This is a critical concept in the proof of the uniqueness of the constructing measure. ([8]).

Definition 4. *Let $\mathbb{D} = \{D_1, \dots, D_n\}$ be an element of $\mathcal{D}$. Denote*

$$\Gamma(\mathbb{D}) = \{\{x_j\}_{j=1}^m \in X^{[*]} : x_j \in \cup\mathbb{D}, j \leq m,\ D_k \cap \{x_j\}_{j=1}^n \neq \emptyset,\ k \leq n\}$$

We call the set $\Gamma(\mathbb{D})$ the limited extraction concerning the extraction basis $\mathbb{D} \in \mathcal{D}$. This set consists of finite sets U included in $\cup\mathbb{D} = \bigcup_{k=1}^n D_k$, and for each $k \leq n$, $U \cap D_k \neq \emptyset$. Therefore, m(the cardinality of $\{x_j\}_{j=1}^m$) must not be less than $n = |\mathbb{D}|$.

Proposition 1. *Let μ be a constructive set function (or a constructively k-additive set function for some fixed $k \in \mathbb{N}$), $\mu^{[*]}$ (resp. $\mu^{[\leq k]}$) be the corresponding constructing measure, and τ be the corresponding Möbius transform. Then, for any $\mathbb{D} \in \mathcal{D}$, we have*

$$\tau(\mathbb{D}) = \mu^{[*]}(\varGamma(\mathbb{D})) \quad (\textit{resp.} = \mu^{[\leq k]}(\varGamma(\mathbb{D}))).$$

Let $\mathcal{A}$ be the family of a disjoint finite union of $\varGamma(\mathbb{D})$ ($\mathbb{D} \in \mathcal{D}$). Then, $\mathcal{A}$ is an algebra over $X^{[*]}$ (resp. $X^{[\leq k]}$) and this implies that $\mu^{[*]}$ (resp. $\mu^{[\leq k]}$) is uniquely defined on $(X^{[*]}, \mathcal{B}^{[*]})$ (resp. $(X^{[k]}, \mathcal{B}^{[k]})$).

Proof. For a constructively k-additive set function, the proposition was shown in [8], and we can prove it similarly for a constructive set function. □

Remark. A set function μ is formulaic k-additive ($k \in \mathbb{N}$) iff $\tau(\mathbb{D}) = 0$ for any $\mathbb{D}$ with $|\mathbb{D}| > k$ ([7]). Then, by using Proposition 1, the constructive k-additivity implies the formulaic k-additivity, because $\varGamma(\mathbb{D}) \subset {X^{[k]}}^c$ if $|\mathbb{D}| > k$.

3 Distortion measures and Constructive set functions

A set function μ on $(X, \mathcal{B})$ is a distorted set function iff there exist a finite non-negative measure ν on $(X, \mathcal{B})$ and a function f on $\mathbb{R}^+$ satisfying $f(0) = 0$, such that $\mu(A) = f(\nu(A))$ for each $A \in \mathcal{B}$. We call the function f a distortion function. Monotonicity of the distortion function is often included in the definition of "distorted measure". In such case, μ is a finite monotone measure. A distoted set function is bounded if the distortion function is continuous.

The following properties are relations between (formulaic and constructive) k-additivities and a distorted measure. These are described in [7], [8], [5].

Proposition 2. *Let $(X, \mathcal{B})$ be a measurable space, μ be a distorted set function on $(X, \mathcal{B})$, f be the corresponding distortion function, and ν be the corresponding finite σ-additive measure.*

(a) ([5],[7])
If f is a polynomial of degree k, then μ is formulaic and constructively k-additive.

(b) ([7])
Assume that, for any $t, s \in \{\nu(A) : A \in \mathcal{B}\}$ ($s < t$) and $A \in \mathcal{B}$ with $\nu(A) = t$, there exists $B \subset A$ such that $\nu(B) = s$ (this property is called "strong Darboux property"). Then, if μ is formulaic k-additive (or constructively k-additive), the degree of f must not be more than k.

As an extension of Proposition 2 (a), we have:

Theorem 1. *Let $(X, \mathcal{B})$ be a measurable space, μ be a distorted set function, f be the corresponding distortion function, and ν be the corresponding finite σ-additive measure. Assume that f is an analytic function described by*

$$f(t) = \sum_{j=1}^{\infty} a_j t^j, \quad a_j \in \mathbb{R},$$

and that

$$\sum_{j=1}^{\infty} |a_j|\, \nu(X)^j < \infty.$$

Then μ is constructive.

Proof. We define a measure values of $\mu^{[\leq k]}$

$$\mu^{[\leq k]}(A^{[\leq k]}) = \sum_{j=1}^{k} a_j \nu(A)^j$$

for any $A \in \mathcal{B}$. This defines the set function $\mu^{[\leq k]}$ on $\mathcal{A}$ because each generalized Möbius transform is determined uniquely by using the above values. Let ν^j be the j-th product measure on X^j, and we restrict this set function on $\mathcal{B}^{[\leq j]} \subset \mathcal{B}^j$. Then, we consider a finite sum of signed measures:

$$\mu^{[\leq k]} = \sum_{j=1}^{k} a_j \nu^j.$$

By the assumption of the theorem, the following limit always exists.

$$\mu^{[*]}(A^{[*]}) = \lim_{k \to \infty} \mu^{[\leq k]}(A^{[\leq k]}).$$

Its finite additivity is clear. Then, we have only to prove, for any set sequence $\{D_m\}_{m \in \mathbb{N}} \subset \mathcal{A}$ satisfying $D_m \searrow$ as $m \to \infty$, $\mu^{[*]}(D_m) \searrow 0$ as $m \to \infty$.

Let $\varepsilon > 0$ be an arbitrarily small positive number. There exists N such that

$$n \geq N \Rightarrow \left| \mu^{[*]}(D) - \mu^{[\leq k]}(D \cap X^{[\leq k]}) \right| < \varepsilon$$

for any $D \in \mathcal{A}$, since the above value can not exceed $\displaystyle\sum_{j=N+1}^{\infty} |a_j| \nu(X)^j$. Using the fact that $\mu^{[\leq k]}$ is a finite measure, we have

$$\lim_{m \to \infty} \mu^{[\leq k]}(D \cap X^{[\leq k]}) = 0.$$

Therefore, we have

$$\limsup_{m \to \infty} \left| \mu^{[\leq k]}(D \cap X^{[\leq k]}) \right| \leq \varepsilon.$$

Using the Carathéodory's extension theorem (see for example [9]) we obtain the claim. □

4 Extraction of k-dimensional Elements

Let μ be a constructive set function, and $\mu^{[*]}$ be its constructing measure. We define the k-dimensional element μ_k ($k \in \mathbb{N}$) of μ as

$$\mu_k(A) = \mu^{[k]}(A^{[*]}) := \mu^{[*]}(A^{[*]} \cap X^{[k]}).$$

In numerical analyses, only the values of set functions may be available, then we consider some methods to extract a k-dimensional element by using the values of the set function μ.

The following proposition may be elementary. However, we give its proof since this describes some important aspects of our assertions.

Proposition 3. *Let $(X, \mathcal{B})$ be a measurable space, and we assume that $\mathcal{B}$ is countably generated. Then the following (1) - (3) are equivalent.*

(1) Each one point set $\{x\}$ $(x \in X)$ is measurable.
(2) For any pair $x, y \in X$ $(x \neq y)$, there exists $A \in \mathcal{B}$ such that $x \in A$ and $y \notin A$.
(3) There exists a sequence of finite partitions

$$\{\mathbb{D}_n\}_{n\in\mathbb{N}} = \left\{\{D_j^{(n)}\}_{j=1}^{N(n)}\right\}_{n\in\mathbb{N}}$$

such that
(3-1) $\mathbb{D}_{n+1}$ is a refinement of $\mathbb{D}_n$ for any $n \in \mathbb{N}$.
(3-2) $\mathcal{B} = \sigma\left(\bigcup_{n\in\mathbb{N}} \mathbb{D}_n\right)$.
(3-3) For each pair $x, y \in X$ with $x \neq y$, there exist $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, there exist $j, k \leq N(n)$ $(j \neq k)$ satisfying $x \in D_j$, and $y \in D_k$.

Proof. (1) $\Rightarrow$ (2) Set $A = \{x\} \in \mathcal{B}$, then $x \in A$ and $y \notin A$.

(3) $\Rightarrow$ (1) Fix $x \in X$. For each $n \in \mathbb{N}$ there exists $j(n) \leq N(n)$ such that $x \in D_{j(n)}^{(n)}$. Then, using the assumption (3.3), for each $y \neq x$, $y \notin D_{j(n)}^{(n)}$ for large enough $n \in \mathbb{N}$. This implies that

$$\{x\} = \bigcap_{n\in\mathbb{N}} D_{j(n)}^{(n)},$$

And the right-hand side set is measurable.

(2) $\Rightarrow$ (3) Using the assumption that $\mathcal{B}$ is countably generated, there exist a countable set family $\{A_n\}_{n\in\mathbb{N}}$ satisfying $\mathcal{B} = \sigma\left(\{A_n\}_{n\in\mathbb{N}}\right)$. Define $A_n^{(0)} = A$ and $A_n^{(1)} = A^c$, and set a partition $\varDelta_n$ as follows.

$$\varDelta_n = \left\{\bigcap_{j=1}^{n} A^{i_j} : (i_j)_{j=1}^{n} \in \{0,1\}^n\right\}$$

Several elements in $\varDelta_n$ may be empty according to the above definition. We may assume that there are no empty sets in $\varDelta_n$ by removing empty sets. In any case, $\{\varDelta_n\}_{n\in\mathbb{N}}$ satisfies (3-1) and (3-2).

Assume that (3-3) is not true, that is, there exists a pair $\{x, y\}$ $(x \neq y)$ satisfying that there exists $A \in \varDelta_n$ with $\{x, y\} \subset A$ for any $n \in \mathbb{N}$. Then, we consider a set $\widetilde{X} = X \setminus \{x, y\} \cup \{\alpha\}$, where $\alpha = \{x, y\}$ is defined as a single point.

We replace an element $A \in \Delta_n$ ($n \in \mathbb{N}$) with $\widetilde{A} = A \setminus \{x, y\} \cup \{\alpha\}$ if $\{x, y\} \subset A$. By the assumption, we have $\{x, y\} \subset A$ or $\{x, y\} \cap A = \emptyset$. Therefore, $\left\{\widetilde{\Delta}_n\right\}_{n\in\mathbb{N}}$ is a sequence of partition of $\widetilde{X}$ satisfying (3.1). Let $\widetilde{\mathcal{B}} = \sigma\left(\left\{\widetilde{\Delta}_n\right\}_{n\in\mathbb{N}}\right)$ and set

$$\mathcal{B}' = \left\{\overline{\widetilde{A}} : \widetilde{A} \in \widetilde{\mathcal{B}}, \overline{\widetilde{A}} = \widetilde{A} \setminus \alpha \cup \{x, y\} \text{ if } \alpha \in \widetilde{A},\ \text{otherwise } \overline{\widetilde{A}} = \widetilde{A}.\right\}$$

Then, $\mathcal{B}'$ is a σ-algebra including $\{\Delta_n\}_{n\in\mathbb{N}}$. This implies $\mathcal{B} \subset \mathcal{B}'$ and that $\{x, y\} \subset A$ or $\{x, y\} \cap A = \emptyset$ for any $A \in \mathcal{B}$. This contradicts the condition of (2). $\square$

We have the following extraction theorem using the above proposition.

Theorem 2. *Let $(X, \mathcal{B})$ be a countably generated measurable space, μ be a constructive set function on $(X, \mathcal{B})$, $\mu^{[\leq *]}$ be the corresponding constructing measure, and $\left\{\Delta_n = \left\{D_j^{(n)}\ {}_j\right\}_{j=1}^{N(n)}\right\}_{n\in\mathbb{N}}$ be a sequence of partitions given in Proposition 3. Assume that $\{x\} \in \mathcal{B}$ for any $x \in X$. Then we have*

$$\mu^{[1]}(A^{[*]}) = \mu^{[*]}(A^{[\leq *]} \cap X^{[1]}) = \lim_{n\to\infty} \sum_{j=1}^{N(n)} \mu(D_j^{(n)} \cap A)$$

Proof. Set

$$E_n = \bigcup_{j=1}^{n} (A \cap D_j^{(n)})^{[*]}.$$

Let us consider a sequence of functions $\{f_n\}_{n=1}^{\infty}$ on $X^{[*]}$ defined by

$$f_n(U) = 1_{E_n}(U).$$

Let $U = \{x_\ell\}_{\ell=1}^{L} \in \left(A \cap D_j^{(n)}\right)^{[*]}$ with $|U| > 1$. There exist $N \in \mathbb{N}$ satisfying

$$n \geq N \Rightarrow \{x_1, x_2\} \not\subset D_j^{(n)},\ j \leq N(n).$$

This implies $U \notin E_n$ for any $n \geq N$. Clearly $\{x\} \in \bigcup_{j\leq N(n)} \left(A \cap D_j^{(n)}\right)^{[*]}$ for any $n \in \mathbb{N}$ if $x \in A$. Thus, we have

$$\lim_{n\to\infty} 1_{E_n}(U) = 1_{A\cap X^{[1]}}(U),$$

for any $U \in X^{[*]}$. On the other hand,

$$\int 1_{E_n}(U)\mu^{[*]}(dU) = \sum_{j=1}^{N(n)} \mu^{[*]}(A \cap D_j^{(n)}) = \sum_{j=1}^{N(n)} \mu(A \cap D_j^{(n)}),$$

and

$$\int 1_{A\cap X^{[\leq 1]}}(U)\mu^{[\leq *]}(dU) = \mu^{[1]}(A^{[1]}).$$

All functions are bounded since these are the finite sum of characteristic functions concerning the partition. The bounded convergence theorem, therefore, implies

$$\lim_{n\to\infty}\sum_{j=1}^{N(n)} \mu(A\cap D_j^{(n)}) = \mu^{[1]}(A^{[1]}),$$

And this concludes the proof. □

Example 1. Set $X = [0,1)$ and let λ be the Lebesgue measure on X. We consider the Borel σ-algebra $\mathcal{B}$ on X. Define $\mu(A) = \lambda(A)^2 + \lambda(A)$ then μ is a distorted measure, and this is constructive (constructively 2-additive).

Consider the following sequence of partitions.

$$\{\Delta_n\} = \left\{ \left\{ \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right) \right\}_{j=1}^{2^n} \right\}.$$

This satisfies the conditions (3-1) - (3-3) in Proposition 3.

Let A be an arbitrary measurable set in $(X, \mathcal{B})$.

$$\begin{aligned}
&\sum_{j=1}^{2^n} \mu\left(\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right] \cap A\right) \\
&= \sum_{j=1}^{2^n} \left(\lambda\left(\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right] \cap A\right) + \lambda\left(\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right] \cap A\right)^2\right) \\
&= \lambda(A) + \sum_{j=1}^{2^n} \lambda\left(\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right] \cap A\right)^2.
\end{aligned}$$

$$\begin{aligned}
0 &\leq \sum_{j=1}^{2^n} \lambda\left(\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right] \cap A\right)^2 \\
&\leq \sum_{j=1}^{2^n} \lambda\left(\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right]\right)^2 \\
&= \sum_{j=1}^{2^n} \frac{1}{(2^n)^2} = \frac{1}{2^n} \to 0, \quad \text{as } n\to\infty.
\end{aligned}$$

Then, we have

$$\lim_{n\to\infty}\sum_{j=1}^{2^n} \mu\left(\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right] \cap A\right) = \lambda(A)$$

This example describes one important aspect of Theorem 2.

We can also prove the higher dimensional extractions as follows.

Theorem 3. *Assume the same conditions and set the same notations with Theorem 2. Remark that a finite partition $\mathbb{D}$ belongs to $\mathcal{D}$, and $\mathbb{D}' \subset \mathbb{D}$ implies $\mathbb{D}'$ is included in $\mathbb{D}$ as a finite set family. We define, for $\mathbb{D} = \{D_j\}_{j=1}^N \in \mathcal{D}$ and $A \in \mathcal{B}$*

$$A \cap \mathbb{D} = \{A \cap D_j\}_{j=1}^N .$$

Then we have, for any $k \in \mathbb{N}$,

$$\mu^{[k]}(A^{[*]}) = \mu^{[*]}(A^{[\le *]} \cap X^{[k]}) = \lim_{n\to\infty} \sum_{\mathbb{D}\subset\mathbb{D}_n, |\mathbb{D}|=k} \tau(A \cap \mathbb{D}).$$

Proof. We will give a similar proof with Theorem 2. The critical point is to construct an adequate approximating sequence for $1_{A^{[*]}\cap X^{[k]}}$. We define

$$f_n(U) = \sum_{\mathbb{D}\subset\mathbb{D}_n, |\mathbb{D}|=k} 1_{\Gamma(A\cap\mathbb{D})}(U).$$

First, we prove the pointwise convergence of the above sequence.

Assume that $\mathbb{D} \in \mathcal{D}$ satisfiles $|\mathbb{D}| = k$, then $|A\cap\mathbb{D}| = k$, ($A\cap\mathbb{D} = \{A \cap D_1, \ldots, A \cap D_k\}$) if $A \cap \mathbb{D} \ne \emptyset$. Then $|U| < k$ implies $U \notin \Gamma(A \cap \mathbb{D})$ because $U \cap (A \cap D_j) \ne \emptyset$ for any $j \le k$ if $U \in \Gamma(A \cap \mathbb{D})$. Therefore, $f_n(U) = 0$ for any $n \in \mathbb{N}$.

Let us consider the case $|U| > k$ ($U = \{u_1, \ldots, u_{k'}\}$, $k' > k$). Fix a pair (i, j) ($1 \le i < j \le k'$), there exists $N_{i,j} \in \mathbb{N}$ satisfying

$$n \ge N_{i,j},\ D \in \mathbb{D}_n \Rightarrow \{u_i, u_j\} \not\subset D.$$

This implies that, if $n \ge \max_{1\le i<j\le k'} N_{i,j}$,

$$U \in \mathbb{D} \subset \mathbb{D}_n \Rightarrow |\mathbb{D}| \ge k' > k.$$

and

$$\mathbb{D} \subset \mathbb{D}_n,\ |\mathbb{D}| > k \Rightarrow U \notin \Gamma(A \cap \mathbb{D}).$$

Assume that $|U| = k$ ($U = \{u_1, \ldots, u_k\}$) and $U \cap A^c \ne \emptyset$. By a similar argument to the above, there exists $N \in \mathbb{N}$ such that.

$$n \ge N, D \in \mathbb{D}_n \Rightarrow |\mathbb{D}| \ge k.$$

Then, if $|\mathbb{D}| = k$ ($\mathbb{D} = \{D_1, \ldots, D_k\}$ and $\mathbb{D} \subset \mathbb{D}_n$ ($n \ge N$), there is one-to-one correspondence with the elements of U and the subsets in $\mathbb{D}$. Therefore, sort the elements of U if necessary, we have

$$u_j \in D_j, \quad \text{for each } j = 1, \ldots, k.$$

By the assumption $U\cap A^c \ne \emptyset$, $u_j \notin A\cap D_j$ for some $j \le k$. Hence, $U \notin \Gamma(A\cap\mathbb{D})$, that is, $f_n(U) = 0$ for any $n \ge N$.

Remark that $\{\Gamma(\mathbb{D}) : \mathbb{D} \subset \mathbb{D}_n, |\mathbb{D}| = k\}$ is a partition of $X^{[k]}$. Then, for each $n \in \mathbb{N}$ and any $U \in A^{[*]} \cap X^{[k]}$, there exist $\mathbb{D} \subset \mathbb{D}_n$ satisfying

$$U \in \Gamma(A \cap \mathbb{D}).$$

That is

$$f_n(U) = 1, \text{ for any } n \in \mathbb{N}.$$

Summing up the above arguments, we have

$$1_{A^{[*]} \cap X^{[k]}}(U) = \lim_{n \to \infty} f_n(U).$$

Using Proposition 1, we have

$$\int_{X^{[*]}} f_n(U) \mu^{[*]}(dU) = \sum_{\mathbb{D} \subset \mathbb{D}_n,\ |\mathbb{D}|=k} \mu\left(\Gamma(\mathbb{D})\right). = \sum_{\mathbb{D} \subset \mathbb{D}_n,\ |\mathbb{D}|=k} \tau(\mathbb{D}).$$

and

$$\int_{X^{[*]}} 1_{A^{[*]} \cap X^{[k]}} d\mu^{[*]} = \mu^{[k]}(A^{[k]}).$$

Then the bounded convergence theorem concludes the proof. □

5 Conclusion

We defined a constructive set function as an extension of a constructively k-additive set function. A distorted measure is constructive if the distortion function is analytic and satisfies some additional conditions. We introduced the concept of an extraction space, which is the family of all finite subsets of the original space. A signed measure on the extraction space represents a constructive set function. A k-dimensional element of a set function on the extraction space was defined, and we gave some methods to calculate the values of k-dimensinal element using generalized Möbius transform. Restriction to k-additive set functions is a valuable method to reduce complexity in numerical analysis. Our theorems will give some checking methods to evaluate the influence of higher dimensional sets.

Some equivalent conditions to be constructive for a distortion measure, to consider the α-dimensional element for a set function for non-integer α, these are future problems.

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