Constructive k-additive measure and Decreasing Convergence Theorems

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Abstract. Pan and concave integrals are division-type nonlinear integrals defined on a monotone measure space. These satisfy the monotone increasing convergence theorem under reasonable conditions. However, the monotone decreasing convergence theorem for these integrals was proved only under limited conditions. In this study we consider the k-additive monotone measure on a non-discrete measurable space, and argue this property of Pan and concave integrals on a k-additive monotone measure space.

Keywords: Monotone Measure · Nonlinear integral · k-additivity.

1 Introduction

In the past several decades, various studies have been conducted on monotone measures (fuzzy measures or capacities) and corresponding nonlinear integrals for both practical and theoretical purpose. In several fields, such as game theory ([1]), economics ([2]), and decision theory ([3]), these concepts are used as strong tools to analyze the targets. In most of these analysis, monotone measures are defined on a finite set and the number of degrees of freedom for a monotone measure represents its exponential growth with respect to the cardinal number of data. Some theoretical studies were motivated by the problem of these combinatorial explosions. In this study, we selected the two perspectives for examining this problem, i.e., “k-additivity” and the “convergence theorem.” The former is a concept for the reduction of parameters to determine a monotone measure and the latter is used for the approximation of the integral values of a measurable function with respect to a monotone measure.

k-additivity was defined using the Möbius inversion formula for a non-additive set function defined on a finite set ([4, 5]). It is utilized to express the set function using the correlation values among subset elements of the space. k-additivity is one typical reduction method for the number of these correlations. Consider a monotone measure on a set with cardinality n; thus, there are $2^n - 1$ parameters to determine the set function. However, for a two-additive measure ($k = 2$), we only need $\dfrac{n(n+1)}{2}$ parameters to determine the monotone measure. The definition of k-additivity for general measurable space was introduced by R. Mesiar [6].

For our arguments and calculations, we reconstruct the concept of k-additivity in the next section, which is essentially same with the Mesiar's definition.

In this paper, $(X, \mathcal{B})$ denotes a general measurable space. A set function $\mu : \mathcal{B} \to \bar{\mathbb{R}}$ is a monotone measure if $\mu(\emptyset) = 0$ and $A \subset B \;\Rightarrow\; \mu(A) \leq \mu(B)$. Thus $(X, \mathcal{B}, \mu)$ is called a monotone measure space. Several integrals are defined on this space. In this study, we focus on the Pan integral [7] and the concave integral [8]. These integrals, defined using the basic sum of simple functions, are called "division-type integrals." For a simple function $\sum_{j=1}^{n} a_j 1_{A_j}$, the basic sum is defined by $\sum_{j=1}^{n} a_j \mu(A_j)$. Several complex situations can be derived from the non-additivity of μ. For example, two basic sums are not the same even if their corresponding simple functions are the same. In this study, we consider some convergence theorems for these integrals when the corresponding monotone measure satisfies "k-additivity."

Let $(X, \mathcal{B}, \mu)$ be a monotone measure space, and f be a measurable function on X. We consider a function $\rho(r) = \mu(f \geq r)$. This function is used to define the Choquet, Sugeno, and Shilkret integrals (for example, the Choquet integral $\int^{\mathrm{ch}} f d\mu = \int_0^\infty \rho(r) dr$). Monotone increasing/decreasing convergence theorems are valid for them under some reasonable conditions. Details regarding these integrals and further theoretical investigations were given by J. Kawabe [10]. These integrals were also dealt in [11] (by Klement et al.), they call the function ρ a survival function.

Our target integrals, Pan and concave, satisfy the monotone increasing convergence theorem under certain conditions ([12]). The monotone decreasing convergence theorem is very sensitive toward these integrals, this property was proved in some limited conditions ([12]). There are two conditions for the monotone decreasing convergence theorem for the Pan integral. The first condition is the sub-additivity. When a monotone measure μ is sub-additive, its Pan integral is a linear function with non-negative coefficients ([13]). Using this property, the monotone decreasing convergence theorem for the Pan integral was demonstrated under the condition of continuity of μ from below ([12]). When μ is sub-additive, its Pan integral is identical to its concave integral. Thus the monotone decreasing theorem is also valid for the concave integral under the same conditions. The second condition entails that "the function sequence converges to 0." In this case, the monotone decreasing convergence theorem is valid if μ is continuous at $\emptyset$" ([12]). In this study, we prove the monotone decreasing convergence theorem for the Pan integral under the condition that "the function sequence is uniformly bounded and μ is k-additive." The idea for this proof is not valid for the concave integral; however, if "the function sequence converges to 0," the monotone decreasing convergence theorem becomes valid for the concave integral, under the above conditions (uniformly bounded, and k-additive).

2 Constructive k-additive measure

k-additivity was originally defined for a set-function on a finite set ([4, 5]) and extended for general measure space by R. Mesiar ([6]). In this section we define a k-additive measure constructively, which concept is almost same with Mesiar's definition.

First, we give an example of a three-points set.

Example 1. Let $X = \{a, b, c\}$ and μ be a set function with

$$\mu(\emptyset) = 0, \quad \mu(\{a\}) = \mu_a, \quad \mu(\{b\}) = \mu_b, \quad \mu(\{c\}) = \mu_c,$$

$$\mu(\{a,b\}) = \mu_{ab}, \quad \mu(\{b,c\}) = \mu_{bc}, \quad \mu(\{a,c\}) = \mu_{ac}, \quad \mu(\{a,b,c\}) = \mu_{abc}.$$

We identify ab with $\{a, b\}$, abc with $\{a, b, c\}$, and so on. Then, its Möbius transform $\{\nu_B\}_{B\subset X}$ is given by

$$\begin{aligned}
&\nu_\emptyset = 0, \quad \nu_a = \mu_a, \quad \nu_b = \mu_b, \quad \nu_c = \mu_c,\\
&\nu_{ab} = \mu_{ab} - \mu_a - \mu_b, \quad \nu_{bc} = \mu_{bc} - \mu_b - \mu_c, \quad \nu_{ca} = \mu_{ac} - \mu_c - \mu_a,\\
&\nu_{abc} = \mu_{abc} - \mu_{ab} - \mu_{bc} - \mu_{ac} + \mu_a + \mu_b + \mu_c.
\end{aligned}$$

For any $A \subset X$, we have

$$\mu(A) = \sum_{B\subset A} \nu_B.$$

Let us consider the set spaces

$$X^{(1)} = \{a, b, c\}, \quad X^{(2)} = \{ab, bc, ac\}, \quad X^{(3)} = \{abc\},$$

and define

$$\mu_1 = \nu_a\delta_a + \nu_b\delta_b + \nu_c\delta_c, \quad \mu_2 = \nu_{ab}\delta_{ab} + \nu_{bc}\delta_{bc} + \nu_{ac}\delta_{ac}, \quad \mu_3 = \nu_{abc}\delta_{abc},$$

where δ_p is the Dirac measure with respect to the point p. Then μ_1, μ_2, and μ_3 are signed measures satisfying

$$\mu(A) = \sum_{j=1}^{3} \mu_j(\{B \in X^{(j)} : B \subset A\}).$$

The set function μ is expressed using three measures.

In general, a set function μ of a finite set X with $\mu(\emptyset) = 0$ can be expressed by the Möbius transform $\{\nu_B\}_{B\subset X}$ as follows.

$$\mu(A) = \sum_{B\subset A} \nu_B.$$

When μ is a k-additive measure ($k \le |X|$), that is, $\nu_B = 0$ if $|B| > k$, μ can be expressed using k signed measures on finite set spaces using the same method with the above example.

In this paper, $(X, \mathcal{B})$ denotes a general measurable space, that is, X is a non-discrete set and $\mathcal{B}$ is a σ-algebra of X. For the precise definitions of σ-algebra, measurable function, Lebesgue integral, and other terms used in the measure theory, can be found in [14]. To define k-additivity on a non-discrete monotone measure space, we define the "finite set space" of a general measurable space.

Definition 1. *Let $(X, \mathcal{B})$ be a measurable space and $j \in \mathbb{N}$. The j-set space $X^{(j)}$ of X is defined by*

$$X^{(j)} = \{(x_i)_{i=1}^j : i \neq i' \ \Rightarrow \ x_i \neq x_{i'}\}.$$

We identify $(x_i)_{i=1}^j$ with $(x'_i)_{i=1}^j$ if $(x_i)_{i=1}^j$ is a permutation of $(x'_i)_{i=1}^j$, and the equivalence relation is denoted by $\sim$. For example, when $X = \{a, b, c\}$, $X^{(2)} = \{\{(a,b),(b,a)\}, \{(b,c),(c,b)\}, \{(c,a),(a,c)\}\}$. Compare with the corresponding set space given in Example 1.

On $X^{(j)}$, we consider the natural σ-algebra $\mathcal{B}^{(j)}$ determined by the direct product and the equivalence relation $\sim$.

For $A \in \mathcal{B}$, we define

$$A^{(j)} = \{(x_i)_{i=1}^j \in X^{(j)}, \ x_i \in A \ (\forall i \leq j) \ \}.$$

Then $A^{(j)} \in \mathcal{B}^{(j)}$. We constructively define the non-discrete k-additivity as follows.

Definition 2. *Let $(X, \mathcal{B})$ be a measurable space, $k \in \mathbb{N}$ and μ is a monotone measure on X. μ is a constructive k-additive measure (in short, k-additive measure) if there exists a finite signed measure μ_j on each $X^{(j)}$, $1 \leq j \leq k$, such that*

$$\mu(A) = \sum_{j=1}^k \mu_j(A^{(j)}).$$

Remark 1. A finite signed measure ν can be expressed by two non-negative finite measures ν^+ and ν^- as follows:

$$\nu = \nu^+ - \nu^-, \ \exists S \in \mathcal{B}, \ \nu^+(A) = \nu(A \cap S), \ \nu^-(A) = -\nu(A \cap S^c), \ \forall A \in \mathcal{B}.$$

(We define $|\nu| = \nu^+ + \nu^-$.) This expression is called "Hahn decomposition" [15], and these notations are used in the sequel.

3 Set Operations

In this section, we establish some properties of set operations. First, we add a notation of a set operation on a set space.

Definition 3. *For $j > 1$, $A \in \mathcal{B}$, and $B \in \mathcal{B}^{(j-1)}$, we define*

$$A(\times)B = \{(a_i)_{i=0}^j : \ a_{i_0} \in A, (a_i)_{i \neq i_0} \in B, \ \exists i_0 \leq j\}.$$

Next, we show the following set inclusions.

Proposition 1. *Fix $j \in \mathbb{N}$, let $\{A_\ell\}_{\ell=1}^L$ ($L \in \mathbb{N}$) be a disjoint subfamily of $\mathcal{B}$, and $B \in \mathcal{B}$ be a measurable set. Then, the following inclusions hold.*

(a)

$$\bigcup_{\ell=1}^{L} \left\{ A_\ell^{(j)} \setminus (A_\ell \cap B)^{(j)} \right\} \subset (B^{(j)})^c.$$

(b)

$$\left(\bigcup_{\ell=1}^{L} A_\ell \right)^{(j)} \setminus \left(\bigcup_{\ell=1}^{L} (A_\ell \setminus B) \right)^{(j)} \subset \left(\bigcup_{\ell=1}^{L} (A_\ell \cap B^c) \right) (\times) X^{(j-1)}.$$

Proof. (a) Consider an element

$$(a_i)_{i=1}^j \in \bigcup_{\ell=1}^{L} \left\{ A_\ell^{(j)} \setminus (A_\ell \cap B)^{(j)} \right\}.$$

Then,

$$(a_i)_{i=1}^j \in A_{\ell_0}^{(j)} \setminus (A_{\ell_0} \cap B)^{(j)}$$

for some $\ell_0 \leq L$. We have $a_i \in A_{\ell_0}$ for any $i \leq j$ because $(a_i)_{i=1}^j \in A_{\ell_0}^{(j)}$. Since

$$(a_i)_{i=1}^j \notin (A_{\ell_0} \cap B)^{(j)},$$

we have

$$(a_i)_{i=1}^j \in (B^{(j)})^c.$$

(b) Consider an element

$$(a_i)_{i=1}^j \in \left(\bigcup_{\ell=1}^{L} A_\ell \right)^{(j)} \setminus \left(\bigcup_{\ell=1}^{L} (A_\ell \cap B) \right)^{(j)}.$$

Then, $a_i \in A_{\ell_i}$ ($\ell_i \leq L$) for any $i \leq j$ because $(a_i)_{i=1}^j \in \left(\bigcup_{\ell=1}^{L} A_\ell \right)^{(j)}$. Since $(a_i)_{i=1}^j \notin \left(\bigcup_{\ell=1}^{L} (A_\ell \setminus B) \right)^{(j)}$, there exists $i_0 \leq j$ such that $a_{i_0} \notin A_\ell \cap B$ for any $\ell \leq L$. This implies that $a_{i_0} \in \bigcup_{\ell=1}^{L} A_\ell \setminus B$ and

$$(a_i)_{i=1}^j \in (\bigcup_{\ell=1}^{L} A_\ell \setminus B)(\times) X^{(j)}. \quad \square$$

Proposition 2. *Let $A_1, A_2, \ldots, A_n \in \mathcal{B}$ be measurable sets. Set $C = \bigcup_{i=1}^n A_i$, then there exists a partition $\{B_1, \ldots, B_L\} \subset \mathcal{B}$ of C satisfying*

$$A_i = \bigcup_{\ell : B_\ell \subset A_i} B_\ell.$$

Additionally, for any set of coefficients $\{a_i\}_{i=1}^n$,

$$\sum_{i=1}^n a_i 1_{A_i} = \sum_{\ell=1}^L \left(\sum_{i:B_\ell \subset A_i} a_i \right) 1_{B_\ell}.$$

Proof. For a measurable set $A \in \mathcal{B}$, define

$$A^{[s]} = \begin{cases} A, & s = 1, \\ A^c, & s = 0. \end{cases} .$$

For $(s_1, s_2, \ldots, s_n) \in \{0,1\}^n$, set

$$D(s_1, s_2, \cdots, s_n) = \bigcap_{i=1}^n A_i^{[s_i]}.$$

Then, $\{D(s_1, \cdots, s_n)\}_{(s_1,\cdots,s_n)\in\{0,1\}^n}$ is a finite disjoint family of sets. Set $\{B_\ell\}_{\ell=1}^L = \{D(s_1, \cdots, s_n)\}_{(s_1,\cdots,s_n)\in\{0,1\}^n}$. Then, $\{B_\ell\}_{\ell=1}^L$ satisfies the proposition. □

4 Monotone Decreasing Convergence Theorems

In this section, we state the monotone decreasing convergence theorems, for Pan and concave integrals on a k-additive monotone measure space. First, we define these integrals.

Definition 4. *We define two families* $\mathcal{S}^p$ *and* $\mathcal{S}^c$ *as follows.*

$$\mathcal{S}^p = \{(a_i, A_i)_{i=1}^n, a_i \in [0,\infty), A_i \in \mathcal{B}, n \in \mathbb{N}, \{A_i\}_{i=1}^n \textit{ is a partition of } X\},$$
$$\mathcal{S}^c = \{(a_i, A_i)_{i=1}^n, a_i \in [0,\infty), A_i \in \mathcal{B}, n \in \mathbb{N}, \{A_i\}_{i=1}^n \textit{ is a covering of } X\}.$$

We regard the families $\mathcal{S}^p$ and $\mathcal{S}^c$ as the corresponding families of simple functions using the identification between $(a_i, A_i)_{i=1}^n$ and $\sum_{i=1}^n a_i 1_{A_i}(x)$. For a monotone measure μ and $\varphi = (a_i, A_i)_{i=1}^n \in \mathcal{S}^p$ or $\mathcal{S}^c$, the basic sum $\mu(\varphi)$ is defined by

$$\mu(\varphi) = \sum_{i=1}^n a_i \mu(A_i).$$

Let $\varphi_1 = (a_i, A_i)_{i=1}^{n_1}, \varphi_2 = (b_i, B_i)_{i=1}^{n_2}$ be two elements of $\mathcal{S}^p$ or $\mathcal{S}^c$.

$$\sum_{i=1}^{n_1} a_i 1_{A_i}(x) = \sum_{i=1}^{n_2} b_i 1_{B_i}(x), \quad \forall x \in X$$

does not imply that $\mu(\varphi_1) = \mu(\varphi_2)$. Then, we describe a simple function as a finite sequence of pairs of a coefficient and a measurable set. However, we often consider $\varphi = (a_i, A_i)_{i=1}^n$ as the function

$$\varphi(x) = \sum_{i=1}^n a_i 1_{A_i}(x),$$

when there is no confusion.

Definition 5. *For a non-negative measurable function f on X, The Pan integral $\int^{\mathrm{pan}}$ and the concave integral $\int^{\mathrm{cav}}$ are defined by*

$$\int^{\mathrm{pan}} f d\mu = \sup\{\mu(\varphi) : \varphi \in \mathcal{S}^p, \varphi \le f\},$$
$$\int^{\mathrm{cav}} f d\mu = \sup\{\mu(\varphi) : \varphi \in \mathcal{S}^c, \varphi \le f\}.$$

These definitions and basic properties are described in [7–9].

Remark 2. By the above definitions, we have

$$f \le g \Rightarrow \int^{\mathrm{pan}} f d\mu \le \int^{\mathrm{pan}} g d\mu, \int^{\mathrm{cav}} f d\mu \le \int^{\mathrm{cav}} g d\mu.$$

The following proposition can be easily obtained from the uniform convergence theorem ([12]). The referred article was originally written in Japanese; therefore, we have provided the direct proof.

Proposition 3. *Assume that $\int^{\mathrm{pan}} 1_X d\mu < \infty$ and μ is continuous from below ($A_n \nearrow A \;\Rightarrow\; \mu(A_n) \nearrow \mu(A)$). Let f be a non-negative measurable function satisfying $\int^{\mathrm{pan}} f d\mu < \infty$. Then,*

$$\int^{\mathrm{pan}} (f+\delta) d\mu \searrow \int^{\mathrm{pan}} f d\mu \quad \text{as } \delta \searrow 0.$$

Proof. Fix $\delta > 0$ and let $\varphi \in \mathcal{S}^p$ be an arbitrary simple function satisfying $\varphi(x) \le f(x) + \delta$.

Define $\varphi_\delta(x) = \sum_{i=0}^{n}((a_i - \delta) \vee 0)\ 1_{A_i}(x)$. Then, we have $\phi_\delta \le f$ and

$$\mu(\phi) \le \mu(\phi_\delta) + \sum_{i=1}^{n} \delta\mu(A_i) \le \int^{\mathrm{pan}} f d\mu + \delta \int^{\mathrm{pan}} 1_X d\mu \searrow \int^{\mathrm{pan}} f d\mu.$$

Therefore,

$$\lim_{\delta \searrow 0} \int^{\mathrm{pan}} (f+\delta) d\mu \le \int^{\mathrm{pan}} f d\mu.$$

Using the clear reverse inequality, we have the required equality. □

A k-additive measure satisfies the condition of the above proposition.

Proposition 4. *If μ is k-additive ($k \in \mathbb{N}$),*

$$\int^{\mathrm{pan}} 1_X d\mu \le \int^{\mathrm{cav}} 1_X d\mu < \infty.$$

Proof. Let $\varphi = (a_i, A_i)_{i=1}^n \in \mathcal{S}^a$ be a simple function satisfying $\varphi(x) = \sum a_i 1_{A_i}(x) \leq 1$. By Proposition 2, there exists $\{B_1, \dots, B_L\} \subset \mathcal{B}$ satisfying

$$A_i = \bigcup_{\ell : B_\ell \subset A_i} B_\ell,\ i \leq n$$

and

$$\sum_{i=1}^n a_i 1_{A_i} = \sum_{\ell=1}^L \left(\sum_{i : B_\ell \subset A_i} a_i \right) 1_{B_\ell}.$$

Set $I_\ell = \{i : B_\ell \subset A_i\}$, then the condition $\varphi \leq 1$ implies that

$$\sum_{i \in I_\ell} a_i \leq 1$$

for any $\ell \leq L$. Then,

$$\begin{aligned}
\sum_{i=1}^n a_i \mu_{A_i} &= \sum_{i=1}^n a_i \sum_{j=1}^k \mu_j \left(\left(\bigcup_{\ell : B_\ell \subset A_i} B_\ell \right)^{(j)} \right) \\
&\leq \sum_{i=1}^n a_i \sum_{j=1}^k |\mu_j| \left(\left(\bigcup_{\ell : B_\ell \subset A_i} B_\ell \right)^{(j)} \right) \\
&\leq \sum_{i=1}^n a_i \sum_{j=1}^k |\mu_j| \left(\left(\bigcup_{\ell : B_\ell \subset A_i} B_\ell \right) (\times) X^{(j-1)} \right) \\
&\leq \sum_{j=1}^k \sum_{\ell=1}^L \left(\sum_{i \in I_\ell} a_i \right) |\mu_j| (B_\ell (\times) X^{(j-1)}) \\
&\leq \sum_{j=1}^k \sum_{\ell=1}^L |\mu_j| (B_\ell (\times) X^{(j-1)}) \\
&\leq \sum_{j=1}^k |\mu_j| (X^{(j)}).
\end{aligned}$$

The right hand side is finite and does not depend on φ. Thus we have

$$\left(\int^{\mathrm{pan}} 1_X d\mu \leq \right) \int^{\mathrm{cav}} 1_X d\mu \leq \sum_{j=1}^k |\mu_j| (X^{(j)}) < \infty. \quad \square$$

Theorem 1. *Let $\{f_n\}$ be a decreasing non-negative measurable function on X. Assume that a monotone measure μ on $(X, \mathcal{B})$ is k-additive ($k \in \mathbb{N}$), and $f_n(x) \leq M < \infty$ for any $n \in \mathbb{N}$ and $x \in X$. Set $f = \lim_{n \to \infty} f_n$, then we have*

$$\int^{\mathrm{pan}} f d\mu = \lim_{n \to \infty} \int^{\mathrm{pan}} f_n d\mu.$$

Proof. For an arbitrary $\delta > 0$ and $n \in \mathbb{N}$, set

$$B_n^{(\delta)} = \{x | f_n(x) \leq f(x) + \delta\}.$$

Because the monotone measure μ is k-additive, there exists a signed measure μ_j on $X^{(j)}$ for each $j \leq k$ such that

$$\mu(A) = \sum_{j=1}^{k} \mu_j(A^{(j)}).$$

By the continuity of signed measures $\{\mu_j\}_{j=1}^k$, $B_n^{(\delta)} \nearrow X$ as $n \to \infty$ implies that $\mu(B_n^{(\delta)c}) \searrow 0$ as $n \to \infty$.

Fix an arbitrary $\varepsilon > 0$. Then, there exists $\delta > 0$ satisfying:

$$\int^{\mathrm{pan}} (f + \delta) d\mu \leq \int^{\mathrm{pan}} f d\mu + \varepsilon.$$

For each $n \in \mathbb{N}$, there exists $\varphi_n = (a_{n,i}, A_{n,i})_{i=1}^{N_n}$ satisfying

$$\mu(\varphi_n) \geq \int^{\mathrm{pan}} f_n d\mu - \varepsilon.$$

Set

$$\widetilde{\varphi_n} = \sum a_{n,i} 1_{A_{n,i} \cap B_n^{(\delta)}} \in \mathcal{S}^p,$$

then $\widetilde{\varphi_n} \leq f + \delta$. By the definition of Pan integral, we have

$$\mu(\widetilde{\varphi_n}) \leq \int^{\mathrm{pan}} (f + \delta) d\mu \leq \int^{\mathrm{pan}} f d\mu + \varepsilon.$$

Using Proposition 1,

$$\begin{aligned}
\mu(\varphi_n) - \mu(\widetilde{\varphi_n}) &\leq \sum_i a_{n,i} \{\mu(A_{n,i}) - \mu(A_{n,i} \cap B_n^{(\delta)})\} \\
&= \sum_j \sum_i a_{n,i} \{\mu_j(A_{n,i}^{(j)}) - \mu_j((A_{n,i} \cap B_n^{(\delta)})^{(j)})\} \\
&= \sum_j \{\mu_j(A_i^{(j)} \setminus (A_i \cap B_n^{(\delta)})^{(j)})\} \\
&\leq \sum_j (\sum_i a_i) |\mu_j| (A_i^{(j)} \setminus (A_i \cap B_n^{(\delta)})^{(j)}) \\
&\leq M \sum_j |\mu_j| (B_n^{(\delta)})^{(j)c}).
\end{aligned}$$

The right hand side tends to 0 as $n \to \infty$, and depends only on n. There exists $n_0 \in \mathbb{N}$ such that

$$\mu(\varphi_n) - \mu(\widetilde{\varphi_n}) < \varepsilon$$

for any $n \geq n_0$. Therefore, we have

$$\mu(\varphi_n) \leq \mu(\widetilde{\varphi_n}) + \varepsilon \leq \int^{\text{pan}} f d\mu + 2\varepsilon.$$

This implies that

$$\lim_{n\to\infty} \int^{\text{pan}} f_n d\mu \leq \int^{\text{pan}} f_n d\mu \leq \int^{\text{pan}} f d\mu + 3\varepsilon \searrow \int^{\text{pan}} f d\mu \quad (\varepsilon \searrow 0).$$

Because the reverse inequality is evident, we have the required conclusion. □

We cannot prove the monotone decreasing convergence theorem for concave integral in similar way. However, in the case where $f_n \searrow 0$, we can prove this property.

Theorem 2. *Let $\{f_n\}$ be a decreasing non-negative measurable function on X satisfying $f_n \searrow 0$ as $n \to \infty$. Assume that a monotone measure on $(X, \mathcal{B})$ is k-additive ($k \in \mathbb{N}$) and $f_n(x) \leq M < \infty$ for any $n \in \mathbb{N}$ and $x \in X$. Then,*

$$\lim_{n\to\infty} \int^{\text{cav}} f_n d\mu = 0.$$

Proof. For $\delta > 0$ and $n \in \mathbb{N}$, set:

$$B_n^{(\delta)} = \{x | f_n(x) \leq \delta\} \nearrow X, \quad \text{as } n \to \infty.$$

Because μ is continuous from above, $(A_n \searrow A \Rightarrow \mu(A_n) \searrow \mu(A))$, $\mu\left(\left(B_n^{(\delta)}\right)^c\right) \searrow 0$ as $n \to \infty$.

By Proposition 4 we have $\int^{\text{cav}} 1_X d\mu < \infty$. Then, we have

$$\int^{\text{cav}} \delta 1_X d\mu = \delta \int^{\text{cav}} 1_X d\mu \searrow 0, \quad \delta \searrow 0.$$

Let $\varphi_n = \sum_{i=1}^N a_i 1_{A_i} \in \mathcal{S}^c$ be a simple function satisfying $\varphi_n \leq f_n$. Set

$$\widetilde{\varphi_n} = \sum_i a_i 1_{A_i \cap B_n^{(\delta)}}.$$

Then, we have

$$\mu(\widetilde{\varphi_n}) \leq \delta \int^{\text{cav}} 1_X d\mu.$$

In contrast,

$$\mu(\varphi_n) - \mu(\widetilde{\varphi_n}) = \sum_i a_i \left(\mu(A_i) - \mu(A_i \cap B^{(\delta)})\right) \tag{1}$$

Using some signed measure μ_j on $X^{(j)}$ for each $j \leq k$, μ can be expressed as follows:

$$\mu(A) = \sum_{j=1}^k \mu_j(A^{(j)}), \quad (A \in \mathcal{B}).$$

Therefore,

$$
\begin{aligned}
(1) &= \sum_i a_i \sum_j \left(\mu_j(A_i^{(j)}) - \mu_j((A_i \cap B_n^{(\delta)})^{(j)})\right) \\
&= \sum_i a_i \sum_j \mu_j \left(A_i^{(j)} \setminus (A_i \cap B_n^{(\delta)})^{(j)}\right) \\
&\leq \sum_i a_i \sum_j |\mu_j| \left((A_i^{(j)}) \setminus (A_i \cap B_n^{(\delta)})^{(j)}\right). \qquad (2)
\end{aligned}
$$

By Proposition 2, there exists a partition $\{D_\ell\}_{\ell=1}^L$ such that

$$
\sum_i a_i 1_{A_i} = \sum_\ell \left(\sum_{i\in I_\ell} a_i\right) 1_{D_\ell}, \quad A_i = \bigcup_{\ell \in \widetilde{I}_i} D_\ell,
$$

where $I_\ell = \{i : D_\ell \subset A_i\},\ \widetilde{I}_i = \{\ell : D_\ell \subset A_i\}$. Using Proposition 1, we have

$$
\begin{aligned}
A_i^{(j)} \setminus (A_i \cap B_n^{(\delta)})^{(j)} &= \left(\bigcup_{\ell\in I_i} D_\ell\right)^{(j)} \setminus \left(\bigcup_{\ell \in I_i} (D_\ell \cap B_n^{(\delta)})\right)^{(j)} \\
&\subset \bigcup_{\ell\in I_i} (D_\ell \cap B_n^{(\delta)^c})(\times) X^{(j-1)}
\end{aligned}
$$

for any $i \leq N$. Then, we have

$$
\begin{aligned}
(2) &\leq \sum_i a_i \sum_j \sum_{\ell \in \widetilde{I}_i} |\mu_j| \left((D_\ell \cap B_n^{(\delta)^c})(\times) X^{(j-1)}\right) \\
&= \sum_j \sum_\ell \left(\sum_{i \in I_\ell} a_i\right) |\mu_j| \left((D_\ell \cap B_n^{(\delta)^c})(\times) X^{(j-1)}\right) \\
&\leq M \sum_j |\mu_j| \left(B_n^{(\delta)^c} (\times) X^{(j-1)}\right) \searrow 0.
\end{aligned}
$$

By the above arguments, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$
\mu(\widetilde{\varphi_n}) < \frac{\varepsilon}{2}
$$

for any $n \in \mathbb{N}$, where $\widetilde{\varphi_n}$ depends on δ for each $n \in \mathbb{N}$. By fixing $\delta > 0$, we can select $n_0 \in \mathbb{N}$ that satisfies

$$
\mu(\varphi_n) - \mu(\widetilde{\varphi_n}) < \frac{\varepsilon}{2},
$$

for any $n \geq n_0$. Thus, we found $\varphi_n \in \mathcal{S}^c$ that satisfies

$$
\mu(\varphi_n) = \mu(\widetilde{\varphi}_n) + \mu(\varphi_n) - \mu(\widetilde{\varphi}_n) < \varepsilon
$$

for any $n \geq n_0$. This implies that

$$
\lim_{n\to\infty} \int^{\mathrm{cav}} f_n d\mu = 0. \quad \square
$$

5 Conclusion

In this study, we constructively defined k-additivity for a non-discrete monotone measure space. For this space, we demonstrated the monotone decreasing convergence theorem for Pan integrals under the condition that the function sequence is uniformly bounded. Furthermore, we demonstrated the monotone decreasing convergence theorem for concave integrals under an additional condition that "the function sequence converges to zero." We believe that these properties can play important role in some arguments of functional analyses where these non-linear integrals are used.

References

1. Fujimoto, K.: Cooperative Game as Non-Additive Measure, Studies in Fuzziness and Soft conputing **310** 131–172 (2014)
2. Ozaki, H.: Integral with Respect to Non-additive Measure in Economics, Studies in Fuzziness and Soft computing, **310** 97–130 (2014)
3. Grabisch, M, Alternative Representations of Discrete Fuzzy Measures for Decision Making, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems **5** (5) 587–607 (1997)
4. Grabisch, M: k-order additive discreet fuzzy measures and their representation, Fuzzy Sets and Systems **92**, 167–189 (1997)
5. Combarro, E.F., Miranda, P.: On the structure of the k-additive fuzzy measures, Fuzzy Sets and Systems **161** (17), 2314–2327 (2010)
6. Mesiar, R.: Generalizations of k-order additive discrete fuzzy measures, Fuzzy Sets and Systems **102** 423-428 (1999)
7. Yang, Q. : The pan-integral on the fuzzy measure space, Fuzzy Mathematica (in Chinese), **3** 107–114 (1985)
8. Lehrer, E., Teper, R.: The concave integral over large spaces, Fuzzy Sets and Systems, **159**, 2130-2144, 2008.
9. Even, Y., Lehrer, E.: Decomposition-integral: unifying Choquet and the concave integrals, Econ Theory **56**, 33-58 (2014)
10. Kawabe, J.: A unified approach to the monotone convergence theorem for nonlinear integrals, Fuzzy Sets and Systems **304** 1–19 (2016)
11. Klement, E. P., Li, J., Mesiar, M., Pap, E.: Integrals based on monotone set functions, Fuzzy Sets and Systems **281** (2015) 88-102
12. Fukuda, R., Honda, A., Okazaki, Y.;Comparison of Decomposition type Nonlinear Integrals Based on the Convergence Theorem (accepted, in Japanese), Journal of Japan Society for Fuzzy Theory and Intelligent Informatics (2020)
13. Ouyang, Y., Li, J., Mesiar,R.: On linearity of pan-integral and pan-integrable functions space, International Journal of Approximate Reasoning archive **90** Issue C 307-318 (2017)
14. Terence. Tao, T.: An Introduction to Measure Theory, Graduate Studies in Mathematics Vol. 126, American Mathematical Society (2011)
15. Billingsley, P.: Probability and Measure,Wiley, 1995