An axiomatic definition of non-discrete Möbius transform

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Abstract. Some global analysis for Möbius transforms must be useful in providing certain analysis tools. It may provide some approximation methods, for example, in the analysis of discrete set functions including numerical analysis. A constructive set function was defined as a set function defined on nondiscrete space, the corresponding generalized Möbius transform can be represented as a signed measure. We defined the covering measure for this set function, which dominates the set function value. This set function is a non-negative subadditive set function, and the corresponding function spaces have rich topological properties.
On the other hand, there are several approaches to generalize the Möbius transform. We attempt to summarize several of them to provide an axiomatic definition. Using this formulation, we defined the notion of bounded variation for the generalized Möbius transform and provide a covering measure using generalized Jordan decomposition of the Möbius transform.

Keywords: fuzzy measure · nonadditive measure · Möbius transform · covering measure

1 Introduction

In some numerical analysis, set functions are approximated using Möbius transform [1]. In such cases, the number of parameters often becomes too large. One useful method to address this issue is to assume k-additivity for the set function [2]. Additionally, non-discrete analysis can be beneficial in resolving such problems. Our study in this paper is motivated by this purpose.

Consider a set X on which a setfunction is defined. Several topologies are discussed for spaces of functions defined on X (for example, [3, 4]). We assume the monotonicity or quasi-monotonicity for μ in such cases. Moreover, various analyses often assume the null additivity, the weak null additivity, or certain continuities. We consider these assumptions too strict to analyze general set functions. The assumptions above imply that any nullsets have no influence on other sets. Section 2 describes these details.

We used the Möbius base function, a generalized Möbius transform defined in [5], to describe the above situations. Denneberg ([13]) defined generalized Möbius transforms as additive set functions defined on the specific spaces of set functions. The constructing measure is similar to the set above functions. [6, 7]. Moreover, Mesiar defined generalized k-additivity using the k-th product space, which is the domain of the corresponding Möbius transform ([2]). In Section 3, we unify these concepts using an axiomatic definition, the Möbius transform as an additive set function on a generalized space.

When μ is a constructive set function, a constructively k-additive ($k \in \mathbb{N}$) function is a typical example. We can define the covering measure $\overline{\mu}$ using the Jordan decomposition of the constructing measure. ([8] for Jordan decomposition.) The covering measure is subadditive, and the corresponding L_p-spaces ($p \geq 1$) are Banach spaces [9]. A measurable set $A \in \sigma(\mathcal{A})$ is a strong μ-nullset if and only if $\bar{\mu}(A) = 0$. Generally, subadditivity or the p.g.p. condition is useful for functional analysis. The p.g.p condition is an equivalence condition to exist an equivalent pseudo metric. Property (S) ([10]) is an essential property for the proof of completeness of some function space. These details are described in [3].

For a monotone set function, Murofushi and Sujino [11] defined the supremum increment (in 2017, before we defined the covering measure), This satisfies properties similar to those of the covering measure.

The constructing measure is a generalized Möbius transform under the definition used in this study. The constructing measure is a σ-additive measure defined on a σ-algebra. The constructing measure can be constructed under bounded variation and fine continuity at $\emptyset$ ([6, 7]). Section 4 defines the Jordan decomposition of our Möbius transform with the bounded variation, in this case, we can define the covering measure using this decomposition.

2 Möbius base function and restrictions by topological assumptions

Let $(X, \mathcal{A})$ be a pre-measurable space, that is, X is a set and $\mathcal{A}$ is an algebra over X. We assume that each set function μ defined on $\mathcal{A}$ satisfies $\mu(\emptyset) = 0$. We call the triplet $(X, \mathcal{A}, \mu)$ a set function space as an analogy of the word "measure space".

In this section, we define the Möbius base function, by which we construct an abstract Möbius transform.

First, we define its domain.

Definition 1. *Let A be an element of $\mathcal{A}$. We define the space of local partitions of A as follows.*

$$\mathcal{D}(A) = \left\{ \{D_j\}_{j=1}^{n} \subset \mathcal{A} : n \in \mathbb{N}, D_j \subset A, \text{ these are pairwise disjoint.} \right\}.$$

This may be shortly denoted by $\mathcal{D}$ if $A = X$. For an element $\mathbb{D} = \{D_j\}_{j=1}^n \in \mathcal{D}(A)$, we define

$$|\mathbb{D}| = n, \quad \cup_{\mathbb{D}} = \bigcup_{D \in \mathbb{D}} D \quad \left(= \bigcup_{j=1}^{n} D_j \right).$$

Let $\mathbb{D}_1 = \left\{D^{(1)}{}_j\right\}_{j=1}^n$, $\mathbb{D}_2 = \left\{D^{(2)}{}_j\right\}_{j=1}^m$ be elements of $\mathcal{D}(A)$, then we define their common refinement:

$$\mathbb{D}_1 \oplus \mathbb{D}_2 = \left\{ D_j^{(1)} \cap D_k^{(2)} : j \leq n, k \leq m \right\} \in \mathcal{D}(A).$$

We define a binary relation $\ll$ between $\mathbb{D}_1 = \left\{D^{(1)}{}_j\right\}_{j=1}^n$, $\mathbb{D}_2 = \left\{D^{(2)}{}_j\right\}_{j=1}^m \in \mathcal{D}(A)$ as follows.

$$\mathbb{D}_1 \ll \mathbb{D}_2 \iff \cup_{\mathbb{D}_1} \subset \cup_{\mathbb{D}_2},\ \forall j \leq n, \exists K_j \subset \{1, \ldots, m\} \ s.t.\ D_j^{(1)} = \bigcup_{k \in K_j} D_k^{(2)}.$$

$\mathbb{D}_1, \mathbb{D}_2 \in \mathcal{D} = \mathcal{D}(X)$ are partitions of X if $\cup_{\mathbb{D}_1} = \cup_{\mathbb{D}_2} = X$, then $\mathbb{D}_2$ is a refinement of $\mathbb{D}_1$ if $\mathbb{D}_1 \ll \mathbb{D}_2$. Moreover, any $\mathbb{D}_3, \mathbb{D}_4 \in \mathcal{D}(A)$ ($A \in \mathcal{A}$) satisfy

$$\mathbb{D}_3, \mathbb{D}_4 \ll \mathbb{D}_3 \oplus \mathbb{D}_4.$$

Remark that, in this case, each element of $\mathbb{D}_3$ or $\mathbb{D}_4$ is a finite union of a subfamily of $\mathbb{D}_3 \oplus \mathbb{D}_4$.

Here, we define a relation $\perp$ between two elements $\mathbb{D}_1, \mathbb{D}_2 \in \mathcal{D}(A)$ as follows.

$$\begin{aligned} \mathbb{D}_1 \perp \mathbb{D}_2 \iff & \mathbb{D}_1 = \emptyset \ or\ \mathbb{D}_2 = \emptyset \\ & or\ \exists D_1 \in \mathbb{D}_1 \ s.t.\ D_1 \cap D_2 = \emptyset, \forall D_2 \in \mathbb{D}_2 \\ & or\ \exists D_2 \in \mathbb{D}_2 \ s.t.\ D_1 \cap D_2 = \emptyset, \forall D_1 \in \mathbb{D}_1. \end{aligned}$$

Fix a partition of A and assume that these are atoms, then we can regard the partition as a finite set. When we consider another partition, we have to define a mixed partition to treat them in the same space. The above definition is available for such a situation.

For $\mathbb{D} \in \mathcal{D}$, we define a subfamily $[\mathbb{D}] \subset \mathcal{A}$ as follows.

$$[\mathbb{D}] = \{A \ :\ A \cap D \neq \emptyset, \forall D \in \mathbb{D}, \text{ and } A \subset \cup_{\mathbb{D}}\}.$$

If $\mathbb{D}_1, \mathbb{D}_2 \in \mathcal{D}$ satisfies $\mathbb{D}_1 \perp \mathbb{D}_2$, then $[\mathbb{D}_1] \cap [\mathbb{D}_2] = \emptyset$.

Next, we define the Möbius base function.

Definition 2. *[5, 9] Let μ be a set function defined on $\mathcal{A}$. We define a function τ on $\mathcal{D}$ as follows.*

$$\tau(\mathbb{D}) = \sum_{\mathbb{D}' \subset \mathbb{D}} (-1)^{|\mathbb{D}| - |\mathbb{D}'|} \mu(\cup_{\mathbb{D}'})$$

τ can be defined inductively as follows ([9]).

(a) $\tau(\{D\}) = \mu(D), \quad \forall D \in \mathcal{A}$.
(b) $\tau(\mathbb{D}) = \mu(\cup_{\mathbb{D}}) - \sum_{\mathbb{D}' \subsetneq \mathbb{D}} \tau(\mathbb{D}'), \quad |\mathbb{D}| > 2.$

We call the function τ Möbius base function concerning $(X, \mathcal{A}, \mu)$. The above formula implies the following inversion formula.

$$\mu(\cup_{\mathbb{D}}) = \sum_{\mathbb{D}' \subset \mathbb{D}} \tau(\mathbb{D}').$$

When we consider that all elements of $\mathbb{D}$ are atoms, $\cup_{\mathbb{D}}$ is a n points set ($n = |\mathbb{D}|$) and the above formula is the inversion formula for the classical Möbius transform.

The following lemma is a key property in this study.

Lemma 1. *([5] Proposition 4.)*
Let μ be a set function on $(X, \mathcal{A})$, τ be the corresponding Möbius base function, and $\mathbb{D} \in \mathcal{D}$, $D_1, D_2 \in \mathcal{A}$ ($\cup_{\mathbb{D}} \cap D_1, D_2 = \emptyset, D_1 \cap D_2 = \emptyset$). Then,

$$\tau(\mathbb{D} \cup \{D_1 \cup D_2\}) = \tau(\mathbb{D} \cup \{D_1\}) + \tau(\mathbb{D} \cup \{D_2\}) + \tau(\mathbb{D} \cup \{D_1, D_2\}).$$

Next, we give some definitions. These may be different from ordinary ones, since we do not assume the monotonicity or quasi-monotonicity for a set function.

Definition 3. *Let $(X, \mathcal{A}, \mu)$ be a set function space and A be an element of $\mathcal{A}$. We define that:*

(a) *A is μ-nullset iff $\mu(A) = 0$.*
(b) *A is μ-c-nullset (μ complete nullset) iff $\mu(B) = 0$ for any $B(\in \mathcal{A}) \subset A$.*
(c) *A is μ-s-nullset (μ strong nullset) iff $\mu(B \cup A) = \mu(B)$ for any $B \in \mathcal{A}$.*

(The name of a set function μ can be omitted if there is no confusion.) Using these concepts, we define:

(d) *μ is monotone iff $A \subset B$ ($A, B \in \mathcal{A}$) implies $\mu(A) \leq \mu(B)$.*
(e) *μ is (strongly) nulladditive iff every c-nullset is a strong nullset.*
(f) *μ is weakly nulladditive iff, for any pair A, B of c-nullsets, $A \cup B$ is a c-nullset.*

Remark 1. Under the assumption of monotonicity, all nullsets are c-nullsets and (e) and (f) can be defined by:

(e)' μ is (strongly) nulladditive iff $\mu(A) = 0$ implies that $\mu(B \cup A) = \mu(B)$ for any $B \in \mathcal{A}$.
(f)' μ is weakly nulladditive iff $\mu(A) = \mu(B) = 0$ ($A, B \in \mathcal{A}$) implies $\mu(A \cup B) = 0$.

For specific analysis targets in cooperative game theory, people may expect something that is born from a cooperative relationship. However, the following properties suggest that, given the assumption of monotonicity or weak or strong null additivity, such utilities cannot be anticipated among c-nullsets.

Lemma 2. *Let μ be a set function on an algebra $\mathcal{A}$. Then:*

(a) *A is a c-nullset iff $\tau(\mathbb{D}) = 0$ for any $\mathbb{D} \in \mathcal{D}(A)$.*
(b) *μ is nulladditive iff*
$\tau(\mathbb{D}) = 0$ if one of $D \in \mathbb{D}$ is a c-nullset.
(c) *μ is weakly nulladditive iff*
$\tau(\mathbb{D}) = 0$ if any element $D \in \mathbb{D}$ is a c-nullset for any $D \in \mathbb{D}$.

Proof. **(a)** clear.

(b)$\Rightarrow$ Assume that $D \in \mathbb{D}$ is a c-nullset and $\tau(\mathbb{D}) \neq 0$.

$$m = \min_{D \in \mathbb{D}', \tau(\mathbb{D}') \neq 0. \mathbb{D}' \subset \mathbb{D}} |\mathbb{D}'|$$

Then $m \leq |\mathbb{D}| < \infty$. Then, without loss of generality, we may assume that $|\mathbb{D}| = m$. Under this condition, $\mathbb{D}' \subsetneq \mathbb{D}$, $D \in \mathbb{D}'$ implies $\tau(\mathbb{D}') = 0$ by the definition of m. Hence, we have:

$$\begin{aligned}
\mu(\cup_{\mathbb{D}}) &= \sum_{\mathbb{D}' \subset \mathbb{D}} \tau(\mathbb{D}') \\
&= \tau(\mathbb{D}) + \sum_{\mathbb{D}' \subsetneq \mathbb{D}, D \in \mathbb{D}'} \tau(\mathbb{D}') + \sum_{\mathbb{D}' \subset \mathbb{D} \setminus \{D\}} \tau(\mathbb{D}') \\
&= \tau(\mathbb{D}) + \mu(\cup_{\mathbb{D} \setminus \{D\}}) \\
&= \tau(\mathbb{D}) + \mu(\cup_{\mathbb{D}} \setminus D)
\end{aligned}$$

This implies that μ is not nulladditive.

($\Leftarrow$) Assume that any c-nullset D and $\mathbb{D} \in \mathcal{D}$ with $D \in \mathbb{D}$ always satisfy $\tau(\mathbb{D}) = 0$.

Let D be a c-nullset and $A \in \mathcal{A}$ be any element. Then, we may assume that $D \cap A = \emptyset$, replacing D by $D \setminus A$ if necessary, to prove $\mu(A) = \mu(A \cup D)$. By the above assumption, we have$\tau(\{D, A\}) = \tau(\{D\}) = 0$. Thus, we have

$$\mu(D \cup A) = \tau(\{D\}) + \tau(\{D, A\}) + \tau(\{A\}) = \tau(\{A\}) = \mu(A).$$

(c) ($\Rightarrow$) Let $\mathbb{D} \in \mathcal{D}$ be a local partition satisfying that every $D \in \mathbb{D}$ is a c-nullset. Then, we have inductively that.

$$\mu(\cup_{\mathbb{D}'}) = 0, \quad \forall \mathbb{D}' \subset \mathbb{D}.$$

Thus, we have

$$\tau(\mathbb{D}) = \sum_{\mathbb{D}' \subset \mathbb{D}} (-1)^{|\mathbb{D}| - |\mathbb{D}'|} \mu(\cup_{\mathbb{D}'}) = 0.$$

($\Leftarrow$) Let D_1 and D_2 be c-nullsets, then D_1 and $D_2 \setminus D_1$ are disjoint. By the assumption $\tau(\{D_1\}) = \tau(\{D_2\}) = \tau(\{D_1, D_2\}) = 0$. Thus

$$\mu(D_1 \cup D_2) = \tau(\{D_1\}) + \tau(\{D_2\}) + \tau(\{D_1, D_2\}) = 0.$$

$\square$

3 Axiomatic definition of Möbius transform

We try to construct a generalized structure for the Möbius transform in this study. There are several non-discrete Möbius transforms, [12, 7, 2, 5] and these Möbius transforms are functions on different domains. (Some of them were given along with k-additivity.) The following axiomatic definition may describe some common properties of the domains of Möbius transform.

Definition 4. *Let $(X, \mathcal{A}, \mu)$ be a set function space, and τ be the corresponding Möbius base function on $\mathcal{D}$. We consider a pair (X^*, Γ), where X^* is a set, and Γ is a mapping from $\mathcal{D}$ to 2^X. Then, we define that (X^*, Γ) is a Möbius field with respect to $(X, \mathcal{A}, \mu)$ iff it satisfies following (M-1)-(M-4).*

(M-1) $\Gamma(\emptyset) = \emptyset, \quad \Gamma(\{X\}) = X^*.$

(M-2) *Assume that $\mathbb{D} \in \mathcal{D}$, $D_1, D_2 \in \mathcal{A}$ satisfy* $(\cup_{\mathbb{D}}) \cap D_1 = (\cup_{\mathbb{D}}) \cap D_2 = \emptyset,\ D_1 \cap D_2 = \emptyset$, *then*

$$\Gamma(\mathbb{D} \cup \{D_1 \cup D_2\}) = \Gamma(\mathbb{D} \cup \{D_1\}) \cup \Gamma(\mathbb{D} \cup \{D_2\}) \cup \Gamma(\mathbb{D} \cup \{D_1, D_2\})$$

(M-3) $\mathbb{D}_1, \mathbb{D}_2 \in \mathcal{D}, \mathbb{D}_1 \perp \mathbb{D}_2 \ \Rightarrow\ \Gamma(\mathbb{D}_1) \cap \Gamma(\mathbb{D}_2) = \emptyset.$

(M-4) $\Gamma(\mathbb{D}) = \emptyset \ \Rightarrow \quad \tau(\mathbb{D}) = 0$, *for any* $\mathbb{D} \in \mathcal{D}$.

Remark that, on the right-hand side of (M-2), the three sets $\Gamma(\mathbb{D} \cup \{D_1\})$, $\Gamma(\mathbb{D} \cup \{D_2\})$ and $\Gamma(\mathbb{D} \cup \{D_1, D_2\})$ are pairwise disjoint by the assumption (M-3).

The following example is the case that the set function is two-additive, that is, all the elements of the Möbius transform elements corresponding to the set with more than two elements are zero. In this case, the structure of the Möbius field is simple.

Example 1. Set $X = \{p : p \in \mathbb{N}, p \leq N\}$, $(N \in \mathbb{N})$, $\mathcal{A} = 2^X$, and $\mu(A) = |A|^2$, $(A \in \mathcal{A})$. Then $(X, \mathcal{A}, \mu)$ is a set function measurable space and μ is a two-additive set function on $\mathcal{A}$ ([5]). Set $X^* = \{A \subset X \mid |A| \leq 2\}$. We define the mapping Γ as follows.

$$\Gamma(\{D\}) = \{\{x\}, \{x, y\} : x, y \in D\}$$
$$\Gamma(\{D_1, D_2\}) = \{\{x, y\} : x \in D_1, y \in D_2\}$$
$$\Gamma(\mathbb{D}) = \emptyset \text{ if } |\mathbb{D}| > 2$$

Then (M-1) is clear. To prove (M-2), we have only to consider the case that $\mathbb{D} = \emptyset$ since $\Gamma(\mathbb{D}) = \emptyset$ if $|\mathbb{D}| > 2$.

When $\{x\} \in \Gamma(\{D_1 \cup D_2\})$, $x \in D_1$ or $x \in D_2$, that is, $\{x\} \in \Gamma(\{D_1\})$ or $\{x\} \in \Gamma(\{D_2\})$. When $\{x, y\} \in \Gamma(\{D_1 \cup D_2\})$, $x, y \in D_1$ or $x, y \in D_2$ or ($x \in D_1$ and $y \in D_2$) or ($x \in D_2$ and $y \in D_1$), that is, $\{x, y\} \in \Gamma(\{D_1\})$ or $\{x, y\} \in \Gamma(\{D_2\})$ or $\{x, y\} \in \Gamma(\{D_1, D_2\})$. Thus, we have

$$\Gamma(\{D_1 \cup D_2\}) = \Gamma(\{D_1\}) \cup \Gamma(\{D_2\}) \cup \Gamma(\{D_1, D_2\}).$$

Next, we prove (M-3). Set $\mathbb{D}_1 = \{D\}, \mathbb{D}_2 = \{D_1, D_2\}$. Then $\mathbb{D}_1 \perp \mathbb{D}_2$ iff

$$D \cap D_1 = \emptyset, \text{ or } D \cap D_2 = \emptyset.$$

Remark that all elements in $\Gamma(\mathbb{D}_2)$ are two-point sets. Let $\{x, y\}$ be an arbitrary element of $\Gamma(\mathbb{D}_2)$. Then, we have $x \notin D$ or $y \notin D$ under our assumption of $\mathbb{D}_1 \perp \mathbb{D}_2$. Thus $\Gamma(\mathbb{D}_1) \cap \Gamma(\mathbb{D}_2) = \emptyset$. In the case where $\mathbb{D}_1 = \{D_1, D_2\}, \mathbb{D}_2 = \{D_3, D_4\}$, $\mathbb{D}_1 \perp \mathbb{D}_2$ implies that

$$(D_1 \cap D_3 = D_1 \cap D_4 = \emptyset) \text{ or } (D_2 \cap D_3 = D_2 \cap D_4 = \emptyset) \text{ or}$$

$$(D_1 \cap D_3 = D_2 \cap D_3 = \emptyset) \text{ or } (D_1 \cap D_4 = D_2 \cap D_4 = \emptyset)$$

Let us consider the first case $(D_1 \cap D_3 = D_1 \cap D_4 = \emptyset)$. Assume that $\{x, y\} \in \Gamma(\mathbb{D}_1)$ and $x \in D_1$. We have $x \notin D_3,\ x \notin D_4$, then $\{x, y\} \notin \Gamma(\mathbb{D}_2)$. Iterate similar arguments three more times, we have $\Gamma(\mathbb{D}_1) \cap \Gamma(\mathbb{D}_2) = \emptyset$.

Then, (X^*, Γ) is a Möbius field with respect to $(X, \mathcal{A}, \mu)$.

In general, if a set function μ satisfies formulaic k-additivity ([5]), we can similarly define a Möbius field with $X^* = \{A : |A| \leq k\}$ ([9, 7]).

Example 2. Let $(X, \mathcal{A})$ be a pre-measurable space. Assume that X is an infinite set, $\mathcal{A}$ is an algebra over X, and μ is a set function on $\mathcal{A}$. Set $X^* = \{A \subset X : |A| < \infty\}$, and for a local partition $\mathbb{D} = \{D_j\}_{j=1}^n$,

$$\Gamma(\mathbb{D}) = \{A \in X^* : A \subset \cup_{\mathbb{D}},\ A \cap D_j \neq \emptyset,\ \forall j \leq n\}$$

Then (X^*, Γ) satisfies (M-1)$\sim$ (M-4), which proof is similar to that of Example 1.

Example 3. Let $(X, \mathcal{A})$ be a same pre-measurable space with Example 2. (X is an infinite set, $\mathcal{A}$ is an algebra over X.) The following spaces were defined in [12].

$$H_p(\mathcal{A}) = \{\eta : \mathcal{A} \to \{0, 1\} \mid \text{ monotone } \eta(\emptyset) = 0, \eta(X) = 1, \text{ supermodular.}\}$$
$$H_u(\mathcal{A}) = \left\{\eta_K : \mathcal{A} \to \{0, 1\} \;\middle|\; K \in \mathcal{A}, \eta_K(A) = \begin{cases} 1 & \text{if } A \supset K, \\ 0 & \text{otherwise.} \end{cases} \right\}$$

$\Gamma(\mathbb{D})$, $(\mathbb{D} \in \mathcal{D}(\mathcal{A}))$ is defined inductively as follows. This structure was essentially given in [12].

(a) $\Gamma(\{D\}) = \widetilde{D} = \{\eta : \eta(D) = 1\}$ (if $|\{D\}| = 1$),

(b) $\Gamma(\mathbb{D}) = \widetilde{\cup_{\mathbb{D}}} \setminus \left(\bigcup_{\mathbb{D}' \subsetneq \mathbb{D}} \Gamma(\mathbb{D}') \right)$ (if $|\mathbb{D}| > 1$).

Then (H_u, Γ) and (H_p, Γ) satisfy (M-1) through (M-4). The proof of this was also essentially provided in [12], and we can similarly demonstrate this using similar methods as in Example 1.

We prepare a notation to describe a property in the next proposition. Let $\mathbb{D}_0, \mathbb{D} \in \mathcal{D}$, and assume that $\mathbb{D} \ll \mathbb{D}_0$. Recall that, in this case, $D \in \mathbb{D}$ implies, for any

$D \in \mathbb{D}$, there exist $\{E_j\}_{j=1}^n \subset \mathbb{D}_0$ s.t. $D = \bigcup_{j=1}^n E_j$. Then, we define the relation $\subset_{\mathbb{D}}$ as follows.

$$\mathbb{D}' \subset_{\mathbb{D}_0} \mathbb{D} \iff \mathbb{D}' \subset \mathbb{D}_0, \cup_{\mathbb{D}'} \subset \cup_{\mathbb{D}} \text{ and}, \forall D \in \mathbb{D}, \exists D' \in \mathbb{D}' \text{ s.t. } D' \subset D.$$

Consider the case where $|X| < \infty$ and $\mathbb{D}_0 = \{\{x_j\}\}_{j=1}^m$. Set $A = \{x_j\}_{j=1}^m$. Then, each element $D' \in \mathbb{D}' \subset \mathbb{D}_0$ is a one-point set, and $A' = \cup_{\mathbb{D}'} \subset A$. We have

$$\mathbb{D}' \subset_{\mathbb{D}_0} \mathbb{D} \iff A' \in \Gamma(\mathbb{D}).$$

In general, $\mathbb{D} \in \mathcal{D}$ is a partition if $\cup_{\mathbb{D}} = X$. Fix this partition and regard each element $D \in \mathbb{D}$ as a single point. We can then define the Möbius transform for the restricted set function. (We define $\mu(D_1, D_2, \cdots, D_k) = \mu(D_1 \cup \cdots \cup D_k)$.) Now, consider another partition $\mathbb{D}'$, which is a refinement of $\mathbb{D}$. We can define another Möbius transform for this refined partition. The following proposition shows that these Möbius transforms are compatible.

Proposition 1. *Let (X^*, Γ) be a Möbius field with respect to $(X, \mathcal{A}, \mu)$, τ be the Möbius base function, and $\mathbb{D}, \mathbb{D}_0 \in \mathcal{D}(X)$ be local partitions satisfying $\mathbb{D} \ll \mathbb{D}_0$. Then we have:*

(a) $\tau(\mathbb{D}) = \sum_{\mathbb{D}' \subset_{\mathbb{D}_0} \mathbb{D}} \tau(\mathbb{D}')$,

(b) $\Gamma(\mathbb{D}) = \bigcup_{\mathbb{D}' \subset_{\mathbb{D}_0} \mathbb{D}} \Gamma(\mathbb{D}')$,

where the union in (b) is a disjoint union.

Proof. We simultaneously prove (a) and (b) by induction over $|\mathbb{D}_0|$. Set $\mathbb{D}_0 = \{D_0\} \in \mathcal{D}$ ($|\mathbb{D}_0| = 1$). Then if $\mathbb{D}' \subset_{\mathbb{D}_0} \mathbb{D}$, it implies $\mathbb{D} = \mathbb{D}' = \{D_0\}$. Therefore, assertions (a) and (b) are evident.

Assume (a) and (b) when $|\mathbb{D}_0| = m$ and consider the case of $|\mathbb{D}_0| = m + 1$. If $\mathbb{D} \subset \mathbb{D}_0$ $(D \in \mathbb{D} \Rightarrow D \in \mathbb{D}_0)$, $\mathbb{D}' \subset_{\mathbb{D}_0} \mathbb{D}$ if and only if $\mathbb{D}' = \mathbb{D}$. Then, our assertions are clear. If $\mathbb{D} \not\subset \mathbb{D}_0$ and $\mathbb{D} \ll \mathbb{D}_0$, there exist $D \in \mathbb{D}$ and $D_1, D_2 \in \mathbb{D}_0$ s.t. $D_0 = D_1 \cup D_2 \subset D$. Set

$$\mathbb{D}_0' = \mathbb{D}_0 \setminus \{D_1, D_2\} \cup \{D_0\}.$$

By the assumption of the induction, we have

$$\tau(\mathbb{D}) = \sum_{\mathbb{D}' \subset_{\mathbb{D}_0'} \mathbb{D}} \tau(\mathbb{D}'),$$

$$\Gamma(\mathbb{D}) = \bigcup_{\mathbb{D}' \subset_{\mathbb{D}_0'} \mathbb{D}} \Gamma(\mathbb{D}').$$

For a local partition $\mathbb{D}' \subset_{\mathbb{D}_0'} \mathbb{D}$ satisfying $D_0 \in \mathbb{D}'$, set $\mathbb{D}'' = \mathbb{D}' \setminus \{D_0\}$. Then, using the Lemma1,

$$\tau(\mathbb{D}') = \tau(\mathbb{D}'' \cup \{D_1\}) + \tau(\mathbb{D}'' \cup \{D_2\}) + \tau(\mathbb{D}'' \cup \{D_1, D_2\}).$$

Using (M-2),

$$\Gamma(\mathbb{D}') = \Gamma(\mathbb{D}'' \cup \{D_1\}) \cup \Gamma(\mathbb{D}'' \cup \{D_2\}) \cup \Gamma(\mathbb{D}'' \cup \{D_1, D_2\}).$$

On the other hand, we have

$$\begin{aligned}\{\mathbb{D}' : \mathbb{D}' \subset_{\mathbb{D}_0} \mathbb{D}\} = \{\mathbb{D}' : \mathbb{D}' \subset_{\mathbb{D}_0'} \mathbb{D}\} \setminus \{\mathbb{D}' : \mathbb{D}' \subset_{\mathbb{D}_0'} \mathbb{D}, D_0 \in \mathbb{D}'\} \\ \cup \{\ \mathbb{D}' \setminus \{D_0\} \cup \{D_1\},\ \mathbb{D}' \setminus \{D_0\} \cup \{D_2\}, \\ \mathbb{D}' \setminus \{D_0\} \cup \{D_1, D_2\},\quad \mathbb{D}' \subset_{\mathbb{D}_0'} \mathbb{D}, D_0 \in \mathbb{D}'\ \}.\end{aligned}$$

Thus, we have proved (a) and (b) for all $\mathbb{D}_0$. □

For a Möbius field (X^*, Γ), $\mathcal{A}^* = \alpha(\{\Gamma(\mathbb{D}) : \mathbb{D} \in \mathcal{D}\})$ (the smallest algebra including $\{\Gamma(\mathbb{D}) : \mathbb{D} \in \mathcal{D}\}$). It is easy to check that

$$\alpha(\{\Gamma(\mathbb{D}) : \mathbb{D} \in \mathcal{D}\}) = \alpha(\{\Gamma(\{D\}) : D \in \mathcal{A}\}).$$

We call the algebra $\mathcal{A}^*$ over X^* the standard algebra over the Möbius field (X^*, Γ).

Theorem 1. *Let (X^*, Γ) is the Möbius field with respect to $(X, \mathcal{A}, \mu)$ and $\mathcal{A}^*$ be the corresponding standard algebra. Then we can define an additive set function τ^* on $\mathcal{A}^*$ satisfying*

$$\tau^*(\Gamma(\mathbb{D})) = \tau(\mathbb{D}) \quad \forall \mathbb{D} \in \mathcal{D}.$$

Proof. By using the condition (M-2), $\mathcal{A}^*$ is a set of all finite disjoint unions of the sets $\Gamma(\mathbb{D})$, $(\mathbb{D} \in \mathcal{D})$. Then, the assertion of the theorem can be followed by Proposition 1. □

4 Möbius transform with Bounded variation

We start by introducing the concept of 'bounded variation.' for a (finitely) additive set function. It's crucial to note that the values of the set function may not always be non-negative.

Definition 5. *Let $(Y, \mathcal{B}, \nu)$ be a set function space. Assume the (finite) additivity on ν. Then, we say ν is of bounded variation iff*

$$\|\nu\| = \sup_{\mathbb{D} \in \mathcal{D}(Y)} \sum_{D \in \mathbb{D}} |\nu(D)| < \infty.$$

We call the non-negative value $\|\nu\|$ the total variation of ν. Moreover, we also define the positive and negative parts.

$$\nu^+(A) = \sup_{\mathbb{D} \in \mathcal{D}(A)} \sum_{D \in \mathbb{D}} \nu(D) \vee 0,$$

$$\nu^-(A) = \sup_{\mathbb{D} \in \mathcal{D}(A)} \sum_{D \in \mathbb{D}} (-\nu(D)) \vee 0.$$

Remark 2. It is evident that $\nu^{\pm}$ are non-negative and monotonic set functions, with $\nu^{\pm}(Y) \leq |\nu| < \infty$ if ν has bounded variation.

The following proposition is a Jordan decomposition for an additive set function with bounded variation.

Proposition 2. *Consider a set function space $(Y, \mathcal{B}, \nu)$. Assuming ν is additive and of bounded variation, then there exist set functions $\nu^{\pm}$ satisfying the following properties:*

(a) $\nu^{\pm}$ *are additive, non-negative, and of bounded variation.*
(b) $\nu(A) = \nu^{+}(A) - \nu^{-}(A)$ *for any* $A \in \mathcal{B}$.

Remark. Under the assumption of the proposition, we can define a non-negative and additive set function $|\nu| = \nu^{+} + \nu^{-}$.

Proof. First, we define $\nu^{\pm}$ on $\mathcal{D}(Y)$ as follows.

$$\nu^{+}(\mathbb{D}) = \sum_{D \in \mathbb{D}} \nu(D) \vee 0, \quad \nu^{-}(\mathbb{D}) = \sum_{D \in \mathbb{D}} (-\nu(D)) \vee 0.$$

Then, we have (double sign in the same order):

$$\mathbb{D} \ll \mathbb{D}' \ \Rightarrow \ \nu^{\pm}(\mathbb{D}) \leq \nu^{\pm}(\mathbb{D}').$$

(a) Let $A, B \in \mathcal{B}$ be disjoint sets ($A \cap B = \emptyset$). Fix an arbitrary $\varepsilon > 0$, then there exist $\mathbb{D}_A \in \mathcal{D}(A)$ and $\mathbb{D}_B \in \mathcal{D}(B)$ s.t.

$$\nu^{+}(A) - \varepsilon \leq \nu^{+}(\mathbb{D}_A) \leq \nu^{+}(A),$$

$$\nu^{+}(B) - \varepsilon \leq \nu^{+}(\mathbb{D}_B) \leq \nu^{+}(B).$$

We have $\mathbb{D}_A \cup \mathbb{D}_B \in \mathcal{D}(A \cup B)$ since $A \cap B = \emptyset$. Moreover,

$$\nu^{+}(\mathbb{D}_A \cup \mathbb{D}_B) = \nu^{+}(\mathbb{D}_A) + \nu^{+}(\mathbb{D}_B)$$

implies

$$\nu^{+}(A) + \nu^{+}(B) - 2\varepsilon \leq \nu^{+}(A \cup B),$$

and, since we can select $\varepsilon > 0$ arbitrally small,

$$\nu^{+}(A) + \nu^{+}(B) \leq \nu^{+}(A \cup B).$$

On the other hand, for any $\varepsilon > 0$, there exist $\mathbb{D}' \in \mathcal{D}(A \cup B)$ s.t.

$$\nu^{+}(A \cup B) - \varepsilon \leq \nu^{+}(\mathbb{D}') \leq \nu^{+}(A \cup B).$$

Set $\mathbb{D}'_A = \{D \cap A : D \in \mathbb{D}'\} \in \mathcal{D}(A)$ and $\mathbb{D}'_B = \{D \cap B : D \in \mathbb{D}'\} \in \mathcal{D}(B)$, then we have

$$\nu^{+}(\mathbb{D}') \leq \nu^{+}(\mathbb{D}'_A) + \nu^{+}(\mathbb{D}'_B) \leq \nu^{+}(A) + \nu^{+}(B),$$

using the subadditivity of ν^{+}. Then, we have

$$\nu^{+}(A \cup B) \leq \nu^{+}(A) + \nu^{+}(B).$$

Since ε is arbitrally small. Thus, we have the additivity of ν^+. The additivity of ν^- can be proved similarly.

(b) Fix an arbitrary $A \in \mathcal{B}$. Then $\exists \mathbb{D}_1, \mathbb{D}_2 \in \mathcal{D}(A)$ satisfying

$$\nu^+(A) - \varepsilon \le \nu^+(\mathbb{D}_1) \le \nu^+(A), \quad \nu^-(A) - \varepsilon \le \nu^-(\mathbb{D}_2) \le \nu^-(A),$$

Set $\mathbb{D} = \mathbb{D}_1 \oplus \mathbb{D}_2$, then $\mathbb{D}_1, \mathbb{D}_2 \ll \mathbb{D}$ and we have

$$\nu^+(A) - \varepsilon \le \nu^+(\mathbb{D}_1) \le \nu^+(\mathbb{D}) \le \nu^+(A),$$
$$\nu^-(A) - \varepsilon \le \nu^-(\mathbb{D}_2) \le \nu^-(\mathbb{D}) \le \nu^-(A).$$

We may assume that $\cup_{\mathbb{D}} = A$, then $\mathbb{D}$ is a partition of A.

$$\nu(A) = \sum_{D \in \mathbb{D}} \nu(D) = \nu^+(\mathbb{D}) - \nu^-(\mathbb{D}).$$

Then,

$$\left|\nu(A) - (\nu^+(A) - \nu^-(A))\right| \le \left|\nu^+(A) - \nu^+(\mathbb{D})\right| + \left|\nu^-(A) - \nu^-(\mathbb{D})\right| \le 2\varepsilon.$$

Therefore, we have proven the assertion.

(c) is clear. □

We call the decomposition $\nu = \nu^+ - \nu^-$ the Jordan decomposition of ν.

Summing up the above arguments, we have the following theorem.

Theorem 2. *Let $(X, \mathcal{A}, \mu)$ be a set function space, (X^*, Γ) be a Möbius field, $\mathcal{A}^*$ be the corresponding algebra over X^*, and τ^* is the corresponding Möbius transform. Assume that τ^* is of bounded variation, then we can define the absolute value $|\tau^*| = \tau^{*+} + \tau^{*-}$ of the Möbius transform τ^*.*

Moreover, we can define a covering measure.

$$\bar{\mu}(A) = |\tau^*|(\Gamma(A) \cup \Gamma(A, A^c)),$$

and $\bar{\mu}$ is subadditive and $A \in \mathcal{A}$ is a strong μ-nullset iff $\bar{\mu}(A) = 0$.

5 Conclusion

In this study, we provided an axiomatic definition of the Möbius transform. The structure of the domain and the proof for the additivity of the Möbius transform can be simplified. We also provided the Jordan decomposition for the Möbius transform, and we obtained the covering measure using it. There were various arguments concerning non-additive set functions or integrals in [12–14]. Our setting is compatible with these studies, and several of them may be improved using the results of this study. Moreover, different types of transforms correspond to the Möbius transform. V. Torra defined the max $\oplus$ transform [15]. Although this is a different type of transform, there are particular relationship with Möbius transform, which is helpful for numerical analysis. Analyzing such transforms is important and represents our future research problems.

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