Measurable Closure of a Finitely Additive Measure Space: An Analysis of Spaces Similar to Stone Spaces

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Abstract. Finitely additive set functions naturally arise in mathematical models, but their extension to σ-additive measures is not always straightforward. For example, a finitely additive measure defined on rational intervals may require space completion to achieve σ-additivity. A common approach involves the use of Stone spaces, which introduce a totally disconnected topology to ensure countable additivity. However, such methods often rely on non-constructive principles. In this paper, we propose a novel and intuitive framework for extending finitely additive set functions by expanding the space, thereby clarifying the relationship between topological and measure-theoretic extensions.

Keywords: finitely additive measure · · σ-additive expansion · Stone space

1 Introduction

Set functions have been used in various forms to describe properties associated with a space. For example, the expected value of a die roll is calculated as 3.5, assuming that each face of the die appears with equal probability. In a more general situation where the corresponding probability measure is not uniform -such as when the die is irregularly shaped- the expected value expressed as an integral can take various values. Furthermore, when the space is an uncountable infinite set, this concept is abstracted and generalized as the Lebesgue integral with respect to a σ-additive measure. In cooperative game theory, allocations are often treated as finitely additive set functions. However, when transitioning to a non-discrete setting and conducting analysis using measure theory, σ-additivity is essential for measure theoretic analysis (see [1, 2] for example).

It is well known that σ-additive measures play a crucial role in various fields, so it need not be stated explicitly here (see, for example, [3]). When modeling an object of analysis mathematically, one may consider defining a finitely additive set function based on observations and intuitive assumptions. When analyzing functions and their properties within integration theory, it is often necessary to extend them to be σ-additive. However, in general, such an extension is not always possible. For example, consider a set function that assigns length to intervals in the space of all rational numbers. A finitely additive set function can be defined on the algebra of sets constructed as finite unions of intervals. However, without changing the total space, this set function cannot be extended to be σ-additive. The reason is that the value of the set function for a single-point set must be 0, and since every subset is countable, if countable additivity held, all values would have to be 0, which contradicts the notion of interval length. In such a case, by completing the space, one can obtain the Lebesgue measure, which is a σ-additive measure. This study analyzes the structure of such extensions.

A well-known approach to achieving such extensions in general is the method using Stone spaces. The Stone space is a concept derived from the Stone Representation Theorem for Boolean algebras. By defining a totally disconnected topology on the space of ultrafilters or certain set functions, it becomes possible to extend finitely additive set functions to be σ-additive. These theories are explained in [4] and [5] . Moreover, [4] employs this method to express the Choquet integral using a σ-additively extended Möbius transformation. The proofs of these results involve elegant arguments utilizing transcendent principles such as Zorn's lemma. (These details are discussed in [5].) Since such extensions involve transcendent aspects, their fundamental nature can be challenging to grasp. In this study, we aim to provide an intuitively comprehensible method for extending finitely additive set functions to be σ-additive by expanding the space. We anticipate that this perspective will elucidate the relationship between extensions of topological and measure-theoretic structures.

2 Cumulative Refinement Space

Let $(X, \mathcal{A}, \mu)$ be a finitely additive measure space, that is, X is a set and $\mathcal{A}$ is an algebra over X, and μ is a nonnegative finitely additive set function defined on $\mathcal{A}$. We assume that μ satisfies $\mu(X) < \infty$.

We define the space of cumulative refinements as follows.

Definition 1. *We define the cumulative refinement space $\mathcal{R} = \mathcal{R}(\mathcal{A})$ as follows.*

$$\mathcal{R} = \mathcal{R}(\mathcal{A}) = \left\{ \mathbb{A} = \{A_n\}_{n\in\mathbb{N}} \middle| A_{n+1} \subset A_n, \forall n \in \mathbb{N} \right\}.$$

For $\mathbb{A} = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{R}$, we define a natural extention $\underline{\mu}$ of μ as follows.

$$\underline{\mu}(\mathbb{A}) = \lim_{n\to\infty} \mu(A_n),$$

where $\mathbb{A} = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{R}$. We define additional notations as follows.

$$\mathbb{A} \cap A = \{A_n \cap A\}_{n\in\mathbb{N}} \in \mathcal{R} \quad \mathbb{A} \in \mathcal{R},\ A \in \mathcal{A},$$
$$\mathbb{A} \cap \mathbb{B} = \{A_n \cap B_n\}_{n\in\mathbb{N}} \in \mathcal{R} \qquad \mathbb{A}, \mathbb{B} \in \mathcal{R}.$$

Example 1. Set $X = [0,1] \cap \mathbb{Q}$, and define

$$\mathcal{A} = \left\{ \bigcup_{j=1}^{n} [a_j, b_j) \middle| n \in \mathbb{N}, a_j, b_j \in X, j \leq n, [a_i, b_i) \cap [a_j, b_j) \text{ if } i \neq j. \right\}$$

and

$$\begin{aligned} \mu([a,b)) &= \mu_1([a,b)) + \mu_2([a,b)) + \mu_3([a,b)) \\ \mu_1([a,b)) &= b - a \\ \mu_2([a,b)) &= \begin{cases} 1 & \text{if } a \leq \frac{1}{2} < b \\ 0 & \text{otherwise.} \end{cases} \\ \mu_3([a,b)) &= \begin{cases} 1 & \text{if } a < \frac{1}{\sqrt{2}} < b \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Define $\mathbb{A}_j = \left\{ [a_n^{(j)}, b_n^{(j)}) \right\}_{n \in \mathbb{N}}$, $(j = 1 \sim 4)$ satisfying

$$\begin{aligned} a_n^{(1)} \nearrow \frac{1}{2},\ b_n^{(1)} \searrow \frac{1}{2}, \quad a_n^{(2)} \nearrow \frac{1}{\sqrt{2}},\ b_n^{(2)} \searrow \frac{1}{\sqrt{2}} \\ a_n^{(3)} \nearrow \frac{1}{3},\ b_n^{(3)} \searrow \frac{1}{3}, \quad a_n^{(4)} \nearrow \frac{1}{\sqrt{3}},\ b_n^{(4)} \searrow \frac{1}{\sqrt{3}.} \end{aligned}$$

Then, we have

$$\underline{\mu}(\mathbb{A}_j) = \begin{cases} 1 & (j=1) \\ 1 & (j=2) \\ 0 & (j=3) \\ 0 & (j=4) \end{cases}, \quad \bigcap_{n \in \mathbb{N}} [a_n^{(j)}, b_n^{(j)}) = \begin{cases} \{\frac{1}{2}\} & (j=1) \\ \emptyset & (j=2) \\ \{\frac{1}{3}\} & (j=3) \\ \emptyset & (j=4) \end{cases}.$$

Thus, elements of $\mathcal{R}$ can fall into four possible categories based on the presence or absence of tangibility (tangible/intangible) and mass (massive/massless).

In the process of finely dividing the whole space X using the elements of the algebra $\mathcal{A}$, the measure of each subset does not necessarily converge to zero.

To analyze such cases, we introduce the following concept.

Definition 2. *We call the element $\mathbb{A} \in \mathcal{R}(\mathcal{A})$ μ-atom if*

(a) $\underline{\mu}(\mathbb{A}) > 0$.
(b) $\underline{\mu}(\mathbb{A} \cap A) = \underline{\mu}(\mathbb{A})$ *or* $\underline{\mu}(\mathbb{A} \cap A) = 0$ *for any* $A \in \mathbb{A}$.

We denote the set of all μ-atoms by $\mathfrak{A}$ and, for each $A \in \mathcal{A}$, $\mathfrak{A}(A)$ denotes the set of all μ-atoms under the restriction to A.

For any $\mathbb{A} \in \mathfrak{A}$ and $\mathbb{B} \in \mathcal{R}$, either $\underline{\mu}(\mathbb{A} \cap \mathbb{B}) = \underline{\mu}(\mathbb{A})$ or $\underline{\mu}(\mathbb{A} \cap \mathbb{B}) = 0$ is true. (By the definition of μ-atom, for any $\mathbb{B} = \{B_n\}_{n \in \mathbb{N}} \in \mathcal{A}$, $\underline{\mu}(\mathbb{A} \cap B_n) = \underline{\mu}(\mathbb{A})$ for any $n \in \mathbb{N}$ or there exists $n \in \mathbb{N}$ satisfying $\underline{\mu}(\mathbb{A} \cap B_n) = 0$. The latter case implies that $\underline{\mu}(\mathbb{A} \cap \mathbb{B}) = 0$ since the sequence $\{B_n\}_{n \in \mathbb{N}}$ is decreasing.)

Definition 3. *Define the relations* $\sim$, $\perp$ *on* $\mathfrak{A}$ *as follows.*

$$\mathbb{A} \sim \mathbb{B} \iff \underline{\mu}(\mathbb{A}) = \underline{\mu}(\mathbb{B}) = \underline{\mu}(\mathbb{A} \cap \mathbb{B})$$
$$\mathbb{A} \perp \mathbb{B} \iff \underline{\mu}(\mathbb{A} \cap \mathbb{B}) = 0,$$

where $\mathbb{A}, \mathbb{B} \in \mathcal{A}$.

Lemma 1. *For the relations* $\sim$, $\perp$ *on* $\mathfrak{A}$ *and* $\mathbb{A}, \mathbb{B} \in \mathfrak{A}$, *we have*

(a) $\mathbb{A} \sim \mathbb{B}$ *or* $\mathbb{A} \perp \mathbb{B}$.
(b) Set $\mathbb{A} = \{A_n\}_{n\in\mathbb{N}}$, $\mathbb{B} = \{B_n\}_{n\in\mathbb{N}}$, *then we have*

$$\mathbb{A} \sim \mathbb{B} \iff \lim_{n\to\infty} \mu(A_n \triangle B_n) = 0.$$

(c) $\sim$ *is a equivalent relation.*
(d) $\mathfrak{A}/\sim$ *is at most countable.*
(e) $\{\underline{\mu}(A) | \mathbb{A} \in \mathfrak{A}\}$ *has the maximal value.*

Proof. (a) $\sim$ (c) are clear.

For the proof for (d) and (e), it is enough to prove, for any $\varepsilon > 0$

$$\{\mathbb{A} \in \mathfrak{A} | \underline{\mu}(\mathbb{A}) > \varepsilon\} / \sim$$

is a finite set. ($\mathbb{A} \sim \mathbb{B}$ implies $\underline{\mu}(\mathbb{A}) = \underline{\mu}(\mathbb{B})$.)

For the assumption in proof by contradiction, we assume that $\{\mathbb{A} \in \mathfrak{A} | \underline{\mu}(\mathbb{A}) > \varepsilon\} / \sim$ is not finite. Then, for n satisfying $\frac{n\varepsilon}{2} > \mu(X)$, there exists $\{\mathbb{A}_j\}_{j=1}^n$ s.t.

$$j, k \le n, j \ne k \Rightarrow \mathbb{A}_j \perp \mathbb{A}_k.$$

and

$$\underline{\mu}(\mathbb{A}_j) > \varepsilon, \quad \forall j \le n.$$

Let $\mathbb{A}_j$ be expressed as $\mathbb{A}_j = \left\{A_k^{(j)}\right\}_{k\in\mathbb{N}}$. Then, for any j, ℓ, we have

$$\lim_{k\to\infty} \mu(A_k^{(j)} \cap A_k^{(\ell)}) = 0.$$

Therefore, for each pair $j, \ell \le n$, there exists large enough $K_{j,\ell} \in \mathbb{N}$ s.t. $\mu\left(A_k^{(j)} \cap A_k^{(\ell)}\right) < \frac{\varepsilon}{2n}$ for any $k \ge K_{j,\ell}$.

$$\begin{aligned}
\mu\left(\bigcup_{j=1}^n A_K^{(j)}\right) &= \sum_{j=1}^n \left(\mu\left(A_K^{(j)}\right) - \sum_{\ell=1}^{j-1} \mu\left(A_K^{(j)} \cap A_K^{(\ell)}\right)\right) \\
&\ge \sum_{j=1}^n \left(\varepsilon - \sum_{\ell=1}^{j-1} \frac{\varepsilon}{2n}\right) \\
&\ge n\varepsilon - \frac{(n-1)\varepsilon}{2} \ge \frac{n\varepsilon}{2} > \mu(X).
\end{aligned}$$

This contradicts the monotonicity of μ and thus concludes the proof. □

In the process of refining the partition of the space, any μ-atom contained in a set is inherited by one of the resulting subsets. Consequently, the measure of such a subset cannot be smaller than the value of the μ-atom.

3 Concepts on finitely additive set functions

First, we introduce the following concept to avoid some unnatural cases.

Definition 4. *Let $(X, \mathcal{A}, \mu)$ be a bounded nonnegative finitely additive set function space.*

(a) $(X, \mathcal{A}, \mu)$ *is natural if and only if there exist* $\mathcal{A}' \subset \mathcal{A}$ *such that* $\mathcal{A} = \mathcal{A}(\mathcal{A}')$ *and* $\mu(A), \mu(A^c) > 0$ *for any* $A \in \mathcal{A}'$*, where* $\mathcal{A}(\mathcal{A}')$ *denotes the smallest algebra containing* $\mathcal{A}'$*.*
(b) $(X, \mathcal{A}, \mu)$ *is separable if it is natural and, for any pair* $x, y \in X$ $(x \neq y)$*, there exist* $A, B \in \mathcal{A}'$ *satisfying* $x \in A,\ y \in B,\ A \cap B = \emptyset$*, where* $\mathcal{A}'$ *is the subfamily of* $\mathcal{A}$ *defined above.*

The following example is a typical non-separable and unnatural set function space.

Example 2. [5] Let X be an infinite set, $\mathcal{A}$ be the set family of all finite or co-finite sets, and μ is defined by

$$\mu(A) = \begin{cases} 1 & |A^c| < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Then, μ is a bounded, non-negative, and finitely additive set function on $(X, \mathcal{A})$. This, however, is neither separable nor natural, which is evident from its properties.

Next, we define $a(A)$, $b(A)$ for each $A \in \mathcal{A}$ as follows.

$$a(A) = \sup\left\{\underline{\mu}(\mathbb{A}) \middle| \mathbb{A} \in \mathfrak{A}(A)\right\}, \quad b(A) = \mu(A) - a(A).$$

We initially attempted to prove the following condition in full generality or under minimal assumptions. However, to avoid unnecessary complications, we chose to incorporate it as an assumption.

Definition 5. *We say that μ is fundamentally decomposable if and only if, for any $A \in \mathcal{A}$ satisfying $b(A) > 0$,*

$$\inf\left\{|\mu(B) - \mu(A \setminus B)|\ |B \in \mathcal{A},\ B \subset A\right\} \leq a(A).$$

In particular, when we assume $a(A) = 0$, for any $\varepsilon > 0$, there exists $B \in \mathcal{A}$ such that $B \subset A$ and $|\mu(B) - \mu(A \setminus B)| < \varepsilon$.

Lemma 2. *Assume that μ is fundamentally decomposable. Then, for each $A \in \mathcal{A}$ satisfying $b(A) > 0$, there exist $B \subset \mathcal{A}$ s.t.*

$$B \subset A,\ b(B), b(A \setminus B) < \frac{4}{5}\, b(A).$$

Proof. In the case where $a(A) = 0$, the lemma is clear by Definition 5.

Assume that $a(A) > 0$. By Lemma 1, $\left\{\underline{\mu}(\mathbb{A}) \middle| \mathcal{A} \in \mathfrak{A}(A)\right\}$ has a maximal value. There exists $\mathbb{A} \in \mathfrak{A}(A)$ such that $\underline{\mu}(\mathbb{A}) = a(A)$. Then, there exists $B_1 \in \mathcal{A}$, satisfying

$$a(B_1) = a(A), \quad a(A) \leq \mu(B_1) < a(A) + \frac{1}{5} b(A),$$

since we assume $b(A) > 0$. Defining $B_2 = A \setminus B_1$, we divide the proof into several cases.

First, we consider the case $a(B_2) < \frac{1}{5} b(A)$. Define

$$\alpha = \inf \left\{ |\mu(D) - \mu(B_2 \setminus D)| \; | D \in \mathcal{A}(B_2) \right\}.$$

Then, the fundamental decomposability of μ implies $\alpha \leq a(B_2)$, and this implies that there exists $D \in \mathcal{A}(B_2)$ satisfying

$$0 \leq \mu(D) - \mu(B_2 \setminus D) < \frac{1}{5} b(A).$$

Remark that the above inequality also implies that $\mu(B_2 \setminus D) \leq \mu(D)$, then we have $\mu(B_2 \setminus D) \leq \dfrac{1}{2}\mu(B_2) < \dfrac{1}{5} b(A)$ using the above assumption.

Define $B = B_1 \cup (B_2 \setminus D)$, then $D = A \setminus B$. Therefore,

$$\begin{aligned}
b(B) &\leq \frac{1}{5} b(A) + \frac{1}{2} \; \mu(B_2) \\
&\leq \frac{1}{5} b(A) + \frac{1}{2} \; b(A) < \frac{4}{5} b(A), \\
b(A \setminus B) &\leq \mu(D) \\
&\leq \frac{1}{2} \; \mu(B_2) + \alpha \\
&\leq \frac{1}{2} \; \mu(B_2) + \frac{1}{5} \; b(A) \\
&\leq \frac{1}{2} \; b(A) + \frac{1}{5} \; b(A) < \frac{4}{5} b(A).
\end{aligned}$$

Next, we consider the case $a(B_2) \geq \frac{1}{5} b(A)$, and define $B = B_1$. Then, we have $A \setminus B = B_2$ and

$$\begin{aligned}
b(B) &\leq \frac{1}{5} b(A) < \frac{4}{5} b(A), \\
b(A \setminus B) &= b(B_2), \\
&< \mu(B_2) - \frac{1}{5} b(A) \\
&\leq b(A) - \frac{1}{5} \; b(A) = \frac{4}{5} \; b(A).
\end{aligned}$$

This concludes the proof. □

4 Measurable Closure

We now define the measurable closure of a finitely additive set function space.

Definition 6. *Let $(X, \mathcal{A}, \mu)$ be a nonnegative, bounded, and finitely additive set function space, $\widetilde{X}$ be a set, and φ is a map from $\mathcal{A}$ to $2^{\widetilde{X}}$. Then, $(\widetilde{X}, \varphi)$ is a measurable closure of $(X, \mathcal{A}, \mu)$ if these satisfy the flowing (a) $\sim$ (e).*

(a) $X \subset \widetilde{X}$, *equivalently, there is an injective map* $\iota : X \to \widetilde{X}$.
(b) $\varphi(\emptyset) = \emptyset$, $\varphi(X) = \widetilde{X}$, *and* $\iota(A) \subset \varphi(A)$ *for each* $A \in \mathcal{A}$.
(c) $A, B \in \mathcal{A},\ A \cap B = \emptyset\ \Rightarrow \varphi(A) \cap \varphi(B) = \emptyset$.
(d) $\forall A, B \in \mathcal{A},\ \varphi(A \cup B) = \varphi(A) \cup \varphi(B)$.
(e) If $A \in \mathcal{A}$ *and* $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ *satisfy* $\varphi(A) = \bigcup_{n=1}^{\infty} \varphi(A_n)$ *then* $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$.

Note that condition (e) is an essential property of this definition.

The following lemma presents a property that can be derived from existing results using standard techniques. Thus, we provide only a guideline for the proof.

Lemma 3. *A measurable closure $(\widetilde{X}, \varphi)$ of $(X, \mathcal{A}, \mu)$ satisfies the following (a) $\sim$ (c).*

(a) $\widetilde{\mathcal{A}} = \{\varphi(A) | A \in \mathcal{A}\}$ *is an algebra.*
(b) $\widetilde{\mu}(\widetilde{A}) = \mu(A)$ *is σ-additive on* $\widetilde{\mathcal{A}}$.
(c) For a subset $\widetilde{A} \subset \widetilde{X}$*,define*

$$\widetilde{\mu}^{\sigma}(\widetilde{A}) = \inf \left\{ \sum_{n=1}^{\infty} \widetilde{\mu}(\widetilde{A_n}) \middle| \left\{\widetilde{A_n}\right\}_{n\in\mathbb{N}} \subset \mathcal{A},\ \widetilde{A} \subset \bigcup_{n=1}^{\infty} \widetilde{A_n} \right\},$$

then,

$$\widetilde{\mathcal{A}}^{\sigma} = \left\{ \widetilde{A} \middle| \widetilde{\mu}(\widetilde{B}) = \mu(\widetilde{B} \cap \widetilde{A}) + \mu(\widetilde{B} \cap \widetilde{A}^c),\ \forall \widetilde{B} \subset \widetilde{X} \right\}$$

is a σ-algebra.
(d) $\widetilde{\mu}^{\sigma}$ *is σ-additive on* $\widetilde{\mathcal{A}}^{\sigma}$*, and* $\widetilde{\mu}^{\sigma}(\widetilde{A}) = \widetilde{\mu}(A)$, $\forall \widetilde{A} \in \widetilde{\mathcal{A}}$.

Outline for the Proof. (a) and (b) are straightforward. The outer measure, defined similarly to $\widetilde{\mu}^{\sigma}$, satisfies sub-additivity and continuity from below. The proof of (c) and (d), therefore, follows from Corollary 2.6 in [6]. □

We also denote the restriction of $\widetilde{\mu}^{\sigma}$ to $\widetilde{\mathcal{B}} = \widetilde{\mathcal{A}}^{\sigma}$ by $\widetilde{\mu}$. The σ-additive measure space $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\mu})$ is also called the measurable closure.

Example 3. Let $(X, \mathcal{A}, \mu)$ be the same set function space with Example 1. $(X = [0,1] \cap \mathbb{Q}.)$ Define

$$\widetilde{X} = [0,1]$$
$$\varphi([a,b) \cap \mathbb{Q}) = [a,b), \quad \forall a,b \in X.$$

Then, we can extend the map φ to $\mathcal{A}$, and $(\widetilde{X}, \varphi)$ is a measurable closure. The extended measure $\widetilde{\mu}$ is given by

$$\widetilde{\mu} = \lambda + \delta_{\frac{1}{2}} + \delta_{\frac{1}{\sqrt{2}}},$$

where λ is the Lebesgue measure on $[0,1]$.

Theorem 1. *Let $(X, \mathcal{A}, \mu)$ be a non-negative, finitely additive, bounded, fundamentally decomposable and separable set function space. Assume that $\mathcal{A}$ is an at most countable family. Then, there exists a measurable closure for $(X, \mathcal{A}, \mu)$.*

Proof. In the case where $\mathcal{A}$ is a finite set, the identity map satisfies the conditions for a measurable closure. Then, we consider the case that $\mathcal{A} \setminus \{\emptyset\} = \{A_n\}_{n \in \mathbb{N}}$ is not finite.

We may assume that $A_1 = X$. We will define a sequence of finite partitions $\left\{\mathbb{D}_m = \left\{D_k^{(m)}\right\}_{k=1}^{N_m}\right\}_{m \in \mathbb{N}}$ ensuring that each $\mathbb{D}_m$ is a refinement of $\mathbb{D}_{m-1}$. The construction proceeds as follows.

Let $\mathbb{D}_1 = \{A_1(= X)\}$, and define $\alpha = \frac{4}{5}$. Set $M = b(A_1)$. Then, it is clear that for any $D \in \mathbb{D}_1$, $b(D) \leq M\alpha^0$ holds. Assume that we have obtained $\mathbb{D}_1 \sim \mathbb{D}_{m-1}$, satisfying the condition that for any $k \leq m-1$ and any $D \in \mathbb{D}_k$, $b(D) \leq M\alpha^{k-1}$. We then set $\mathbb{D}' = \mathbb{D}_{m-1}$. Clearly, for any $D \in \mathbb{D}'$, we have $b(D) \leq M\alpha^{m-2}$.

In the case where $A_m \notin \mathcal{A}(\mathbb{D}')$, where $\mathcal{A}(\mathbb{D}')$ is the smallest algebra containing $\mathbb{D}'$, consider the refinement

$$\{D \cap A_m, D \cap A_m^c | A \in \mathbb{D}'\},$$

and this refinement is also denoted by $\mathbb{D}'$.

In the case where $D \in \mathbb{D}'$ satisfies $M\alpha^{m-1} < b(D)(\leq M\alpha^{m-2})$, we need to subdivide D so that each resulting subset D' satisfies $b(D') \leq M\alpha^{m-1}$. By Lemma 2, there exist $D_1, D_2 \in \mathcal{A}$ such that $D' = D_1 \cup D_2$,

$$b(D_1), b(D_2) \leq \alpha b(D') \leq M\alpha^{m-1}.$$

Thus, by induction, we obtain a sequence of partitions $\{\mathbb{D}_m\}_{m \in \mathbb{N}}$ satisfying, for each $m \in \mathbb{N}$, the condition

$$b(D) \leq M\alpha^{m-1}$$

for all $D \in \mathbb{D}_m$.

Next, we consider the following family of decreasing sequences of sets.

$$\widetilde{X} = \left\{ \mathbb{S} = \left\{ D_{k_m}^{(m)} \right\}_{m\in\mathbb{N}} \,\middle|\, \{k_m\}_{m\in\mathbb{N}} \in \prod_{m\in\mathbb{N}} \{1,2,\cdots,N_m\}, \right.$$
$$\left. D_{k_{m+1}}^{(m+1)} \subset D_{k_m}^{(m)} \ \forall m \in \mathbb{N} \right\} \subset \mathcal{R}.$$

Next, we define $\iota; X \to \widetilde{X}$ as follows. Fix arbitrary $x \in X$ and $m \in \mathbb{N}$. Then, there uniquely exists $j(m,x) \leq N_m$ such that $x \in D_{j(m,x)}^{(m)}$. Define $\iota(x) = \left\{ D_{j(m,x)}^{(m)} \right\}_{m\in\mathbb{N}} \in \widetilde{X}$. Then, as we defined in Definition 4, the separability implies that ι is injective.

From the construction of the partition $\{\mathbb{D}_m\}_{m\in\mathbb{N}}$, for each $A \in \mathcal{A}$, there exist $m \in \mathbb{N}$ and $\mathbb{D}' \subset \mathbb{D}_m$ such that $A = \cup_{\mathbb{D}'}$. We can define

$$\varphi(A) = \bigcup_{D\in\mathbb{D}'} \left\{ \mathbb{A} \in \widetilde{X} \,\middle|\, D \in \mathbb{A} \right\}.$$

and

$$\widetilde{\mathcal{A}} = \{\varphi(A) | A \subset \mathcal{A}\} \cup \{\emptyset\}.$$

Then, we obtain that $\widetilde{\mathcal{A}}$ is an algebra.

Using μ-atoms, we define

$$b'(A) = \mu(A) - \sum_{\mathbb{A}\in\mathfrak{A},\mathbb{A}\subset A} \underline{\mu}(\mathbb{A}),$$

where $\mathbb{A} = \{A_n\}_{n\in\mathbb{N}} \subset A$ means that $A_n \subset A$ for sufficiently large n. Remark that we have $b'(A) \leq b(A)$ for any $A \in \mathcal{A}$.

Using the fact that two different atoms $\mathbb{A}_1, \mathbb{A}_2 \in \widetilde{X}$ satisfy $\mathbb{A}_1 \perp \mathbb{A}_2$ and Lemma 1, we have

$$\mu(A) = b'(A) + \sum_{\mathbb{A}\in\widetilde{X},\mathbb{A}\subset A} \underline{\mu}(\mathbb{A}).$$

Set $I = [0, b'(X)]$ and $\mathfrak{A}' = \left\{ \mathbb{A} \in \widetilde{X} \,\middle|\, \mathbb{A} \in \mathfrak{A} \right\}$. Let λ be the Lebesgue measure on I, and define a σ-additive measure ν on $\mathfrak{A}'$ as follows.

$$\nu(U) = \sum_{\mathbb{A}\in U} \underline{\mu}(\mathbb{A}), \quad U \subset \mathfrak{A}'.$$

We consider the measure space $(I \cup \mathfrak{A}', \lambda + \nu)$. For an arbitraly element $D_j^{(m)}$ $(m \in \mathbb{N},\ j \leq N_m)$, we define

$$\psi(\varphi(D_j^{(m)})) = \left[\bigcup_{k=1}^{j-1} b'(D_k^{(m)}), \bigcup_{k=1}^{j} b'(D_k^{(m)}) \right) \cup \left\{ \mathbb{A} \,\middle|\, \mathbb{A} \subset D_j^{(m)} \right\}$$

Thus, we defined a mapping ψ from $\mathcal{A}'$ to $\mathcal{B}(I) \cup 2^{\mathfrak{A}'}$. By the above definitions, for each $A \in \mathcal{A}$, we have

$$\mu(A) = \widetilde{\mu}(\varphi(A)) = (\lambda + \nu)(\psi(\varphi(A))).$$

Suppose that $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and $A \in \mathcal{A}$ ($\{\varphi(A_n)\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{A}}, \varphi(A) \in \widetilde{\mathcal{A}}$) satisfy

$$\varphi(A) = \bigcup_{n=1}^{\infty} \varphi(A_n),$$

we have

$$\psi(\varphi(A)) = \bigcup_{n=1}^{\infty} \psi(\varphi(A_n)).$$

This equality is key to the proof, and its validity can be shown by assigning to each $\mathbb{A} \in \widetilde{X}$ a Cauchy sequence in I. The sum on the right-hand side is not necessarily a disjoint union, however, the measure of every pairwise intersection is zero. We therefore have

$$(\nu + \lambda)(\psi(\varphi(A))) = \sum_{n=1}^{\infty} (\nu + \lambda)(\psi(\varphi(A_n))),$$

and

$$\widetilde{\mu}(\varphi(A)) = \sum_{n=1}^{\infty} \widetilde{\mu}(A_n).$$

Thus, we have the σ-additivity of $\widetilde{\mu}$ and this concludes the proof. □

Conclusion

In this paper, we introduced a novel approach for extending finitely additive set functions to σ-additive measures through the construction of a measurable closure. Unlike classical methods based on Stone spaces, our framework emphasizes intuitive and constructive techniques that avoid reliance on non-constructive tools such as Zorn's lemma.

By systematically refining partitions and utilizing the structure of μ-atoms, we demonstrated how to embed a given finitely additive measure space into a larger space in which σ-additivity naturally emerges. The construction of a measurable closure allows for a seamless transition from algebraic to topological and measure-theoretic perspectives.

Furthermore, we showed that under assumptions such as separability and fundamental decomposability, the existence of a measurable closure is guaranteed. This provides a concrete path to understand the limitations and possibilities of extending finitely additive measures, shedding new light on the interplay between algebraic structure and measure-theoretic completeness.

We hope that this framework serves not only as a tool for analysis but also as a foundation for further exploration into the constructive aspects of measure theory and its applications in related fields. For the generalized Möbius transform, for example, an axiomatic definition of the Möbius transform was provided in [7], which includes the one defined in [4], and these are formulated as finitely additive set functions. Further developments can be expected based on the analysis presented in this paper.

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